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BOOK: Operation Research by Hira and Gupta –S Chand

Only for Reference

TRANSPORTATION PROBLEM AND VARIANTS

Introduction to Lecture

T: Welcome to the next exercise. I hope you enjoyed the previous exercise.

S: Sure I did. It is good to learn new concepts. I am beginning to like the conversation mode of studying difficult problems.

T: Shall we proceed to the next problem?

S: Yes

T: Consider that a company transports desk top computers from three warehouses to three retail stores. The number of computers available (supply) in the three warehouses are 40 50 and 30 while the number of computers required at the three retailers (demand) are 20, 35 and 65 respectively. There is a cost of transporting from the warehouses to the retail outlets. The cost transport one unit from a given warehouse to a given retailer is given in the Table 2.1. How do we transport the computers such that we incur minimum total cost of the transportation?

Table 2.1- Transportation c osts

9	6	10	40
10	8	9	50
5	4	7	30
20	35	65	

S: The problem looks very interesting. I have a few questions. Should a retailer get all the supplies from a single warehouse?

T: No. Not necessary. In fact, it may not even be possible. For example, the demand of 65 of the third retailer has to be met from at least two warehouses fully or partly.

S: If I consider retailer 1, the requirement is 20. Can I have all of them supplied from a single warehouse?

T: You can – provided it is the least cost way. It is not absolutely necessary. We can meet the 20 units by transporting them from two different warehouses.

S: I also observe that the sum of the requirements is 120 which equal the sum of the computers available in the three warehouses.

T: Very good observation. In order to be able to transport and meet the demands, we should have at least 120 available in the three warehouses. It is good to solve the problem where the total availability is equal to the total requirement.

S: If the availability is more, what do I do?

T: At present let us concentrate on the problem on hand. Other variations can be addressed later

S: Sorry, let me concentrate on the task on hand.

T: First can you give me a solution to the problem. Then we can discuss whether this is the best or can be improved.

S: I will name the three supplies as S1 to S3 and the three retailers as D1 to D3. Let me take the retailers one at a time. I can send 20 from S1 to D1 (and meet the entire demand). I send 35 from S2 to D2 (and meet the entire demand). I have 20 left with S1, 15 with S2 and 30 with S3. I send all these to D3 and meet the demand.

T: Good. You have provided a solution. Have you provided a solution that can be implemented?

S: I think so. I have met the requirements of the three retailers. I have used all the available supplies. Therefore I have provided a solution that can be implemented.

T: Yes. You have provided a solution that can be implemented. It has not violated the supply availability. It has used all the supplies and has met all the demands. Can you compute the cost of this solution?

S: The total cost is 20x9 + 20x10 + 35x8 + 15x9 + 30x7 = 1005.

T: Can you show this solution in the table?

S: Yes. It is shown in Table 2.2

Table 2.2 - Solution

9 *20*	*6*	*10* *20*	*40*
10	*8* *35*	*9* *15*	*50*
5	*4*	*7* *30*	*30*
20	*35*	*65*	

T: You have given a solution with total cost = 945. Can you think of solutions with lesser cost? Before that can we discuss some aspects of your solution?

S: I have tried to meet the demands from a single supplier as much as I can. In fact, I should be able to meet the demand of at least one retailer fully.
T: The easier thing is not to worry about meeting the demand entirely from one supplier. You need not have concentrated on that aspect at all. Instead, why don't you try meeting as much of the demand as possible from any supplier?

S: Let me consider D1 and S1. I can meet the demand of 20 from S1. Now S1 has 20 units available.

T: You can use this 20 to meet the demand of D2 partly.

S: I transport 20 from S1 to D2. I have used all the 40 from S1 and have to meet 15 of D2. I use the supply from S2 to do so. I send 15 from S2 to D2. I have met the entire demand of D2. I have 35 left in S2. I now meet the 65 of D3 by sending 35 from S2 and 30 from S3.

T: Can you show the solution using the table? Please compute the total cost so that we can compare the cost with the 1005 that we arrived at earlier.

Table 2.3 - Solution

9 20	6 20	10	40
10	8 15	9 35	50
5	4	7 30	30

S: Yes. The solution is shown in Table 2.3. The total cost associated with this solution is 20x9 + 20x6 + 15x8 + 35x9 + 30x7 = 945. The cost is now less. This is a better solution than the previous one.

T: Would you agree with me that the effort is also less compared to the earlier case when tried to get as much as we could from a single supplier?

S: I am not sure. I think I took the same time and effort when I computed either of the solutions.

T: The second one would have been faster if you had written a computer program.

S: Is it? It should be true if you say so.

T: Would you like to see if better solutions exist?

S: I don't know how to proceed. I am stuck.

T: Did you realize that in both the solutions, you did not consider the costs when you made the allocations? In your anxiety to get a good solution, you totally forget the costs. You computed the cost after you made all the allocations.

S: I should try to get a solution where I allocate based on minimum cost. The minimum cost is between S2 and D2. I can use the 30 supply from S2 and meet part of the demand of D2. Now S2 has been fully utilized while D2 has an unmet demand of 5.

T: Please continue with this idea.

S: Continuing, I consider the least cost which is between S1 and D2. I meet the demand of 5 from S1 and now S1 has a supply of 35. The next least cost is 9. There is a tie.

T: What about the costs between S3 and D1. Is it not less than 9?

S: I do not consider this because all the supply from S3 has been consumed at this point. I also leave out S3-D3 and S2-D2. Now I have a tie between S2-D3 and S1-D1. I am stuck.

T: You can break the tie by choosing any one arbitrarily.

S: I choose S2-D3 and allocate the maximum of 50, thereby consuming all the supply from S2. I allocate 20 from S1 to D1 and 15 from S1 to D3.

T: Can you show the solution using the table? Please compute the total cost so that we can compare the cost with 945 that we have.

Table 2.4 - Solution

9 20	6 5	10 15	40
10	8	9 50	50
5	4 30	7	30

S: The solution is given in Table 2.4. The total cost associated with this solution is 20x9 + 5x6 + 15x10 + 50x9 + 30x4 = 930. The cost is now less. This is a better solution than the previous one.

T: Are you convinced now that considering costs while making the allocations can give better solutions?

S: Yes. I should have tried this first. This approach should give me the best possible allocation with the least possible cost.

T: Are you sure that this is the best solution with the minimum possible cost?

S: I think so. I have made allocations based on minimum possible cost at each point. It can't get better than this. I should have got the least cost solution.

T: I am not sure. For example, you did not consider the costs while making the last two allocations. The earlier allocations did not leave you with any choice at that point. For example, irrespective of the costs you had to allocate 20 and 15. Therefore your claim that this solution has to be the best is questionable.

S: I am beginning to understand. I do agree that even though I made allocations based on least cost at any point, my choices were restricted towards the end of the allocations. There is a chance that there could be a solution with lesser cost than what I have presented.
T: In this case there is a solution with lesser cost.

S: Please show me the solution and tell me the way by which I can get that solution.

T: I will show you the solution in Table 2.5. You can compute the total cost and verify whether it is lesser.

Table 2.5- Solution.

9	*6* *25*	*10* *15*	*40*
10	*8*	*9* *50*	*50*
5 *20*	*4* *10*	*7*	*30*

S: The total cost associated with this solution is 25x6 + 15x10 + 50x9 + 20x5 + 10x4 = ***890****. The cost is now less. This is a better solution than the previous one. How did you arrive at this solution? Is this the best solution? Can we further reduce the cost?*

T: Before my solution, you gave three solutions. All of them were obtained by repeating simple steps consistently so that all the demand was met. Let us call them simple procedures. Now, there is no single simple procedure by which we can get the best solution for all problem instances of this type. It may be possible to create one that will work for this problem while it may not give the best for some other problem.

S: How do you prove that this is the best solution?

T: There are procedures to get the best solution from any simple solution. For example, if you consider the previous solution with cost = 930 and try to modify it by allocating from S3 to D1 and adjusting the solution carefully, you can get the solution that I have provided. You will learn these methods in this chapter.

S: How will you prove that the cost cannot be reduced further?

T: I didn't tell you why you should consider allocating from S3 to D1. We can compute the increase or decrease in cost if we tried allocations in empty positions. When there is no empty position that can decrease the cost further we can say that the best solution has been obtained.

S: This is a little complicated for me.

T: I agree with you. We will stop this exercise here. You were able to get a very good solution using a simple procedure. Being able to establish the best solution was difficult.

S: Thank you. I look forward to more exercises. I hope I will do better in the next exercise.

The Transportation problem is one of the most important and well addressed problems in Operations Research. The origins of this problem are attributed to Hitchcock (1941) and it was referred to as Hitchcock problem sometimes. It has wide applications in logistics and distribution management. Several special algorithms are available to solve the basic transportation problem and some of its variants. The basic transportation problem is a linear programming problem that can be solved using the excel solver.

The transportation problem is described as follows:

Given a single commodity of which a_i quantity is available in warehouse i, it is required b_j quantities to destination j at minimum total cost. C_{ij} represents the cost of transporting a unit from warehouses i to destination j.

The formulation of the transportation problem is as follows:
Let X_{ij} be the quantity transported from warehouse (supply) i to destination (demand) j. There are m supply points and n destinations.

The objective is to Minimize $\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij} X_{ij}$

Subject to

$$\sum_{j=1}^{n} X_{ij} \leq a_i \ \forall i$$

$$\sum_{i=1}^{m} X_{ij} \geq b_j \ \forall j$$

X_{ij} integer

We have defined X_{ij} to be integers under the condition that we do not wish to transport fractional quantities.

With the above formulation, we will be able to transport the required quantities if $\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$ i.e if the total supply exceeds total demand. A transportation problem where total supply is equal to the total demand is called a balanced transportation problem and we transport all the quantities available in the supply points to all the destination or demand points.

Illustration 2.1

A company that sells TV sets has two warehouses and three four retail outlets. They have to transport 40, 50, and 20 TV sets to the four retailers. The warehouses have 65, and 45 sets. The unit cost of transportation from i to j is given in Table 2.6

Table 2.6 – Unit cost of transportation

	R1	R2	R3
W1	15	20	22
W2	9	18	14

Find the least cost transportation plan?

The total supply is 110 and the total demand is 110. We will be able to meet the demands using these supply quantities. Let X_{ij} be the quantity transported from supply i to destination j. The objective is to

Minimize $15X_{11} + 20X_{12} + 22X_{13} + 9X_{21} + 18X_{22} + 14X_{23}$

Subject to

$X_{11} + X_{12} + X_{13} \leq 65$

$X_{21} + X_{22} + X_{23} \leq 45$

$X_{11} + X_{21} \geq 40$

$X_{12} + X_{22} \geq 50$

$X_{13} + X_{23} \geq 20$

$X_{ij} \geq 0$ and integer.

We formulate this problem using the solver. We include “linear model” and “non negativity” in the options and solve. The solution is given in Table 2.7

x11	x12	x13		15	20	22
x21	x22	x23		9	18	14
15	**50**	**0**				
25	**0**	**20**				
obj =	**1730**					
65	<=	65				
45	<=	45				
40	>=	40				
50	>=	50				
20	>=	20				

The optimum solution to this problem with integer restriction is given by X_{11} =15, X_{12} = 50, X_{21} = 25, X_{23} = 20, with total cost = **1730**. (Table 2.7 does not explicitly show the integer restriction as it is included as constraints in the solver)

We have four variables in the solution.

If we ignore the integer restriction on the variables, the formulation becomes an LP problem. The optimum solution to the LP is the **same.**

Illustration 2.2

Consider the problem of transporting TV sets of a particular brand (TVX) from 2 warehouses to 3 retail shops. The demand in the three retailers is 80, 60 and 90 while 100 and 120 sets are available in the two warehouses. The cost of transportation is defined as a unit cost of transporting an item from a warehouse to retailer and these are given in Table 2.8. Find the optimum solution with least total cost of transportation?

Table 2.8 – Supply, demand and costs for a transportation problem

4	6	5	100
5	7	8	150
80	60	90	

The transportation problem is formulated as a Linear Programming problem as follows:

Minimize $4X_{11} + 6X_{12} + 5X_{13} + 5X_{21} + 7X_{22} + 8X_{23}$
Subject to
$X_{11} + X_{12} + X_{13} \leq 100$
$X_{21} + X_{22} + X_{23} \leq 150$
$X_{11} + X_{21} \geq 80$
$X_{12} + X_{22} \geq 60$
$X_{13} + X_{23} \geq 90$

$X_{ij} \geq 0$ and integer.

We formulate this problem using the solver. We include "linear model" and "non negativity" in the options and solve. The solution is given in Table 2.9

Table 2.9 – Solution to the transportation problem

	x11	x12	x13				
	x21	x22	x23				
	0	10	90		4	6	5
	80	50	0		5	7	8
obj fn	**1260**						
cons	100	<=	100				
	130	<=	150				
	80	>=	80				
	60	>=	60				

	90	>=	90				

The solution is given by X_{12} = 10, X_{13} = 90, X_{21} = 80 and X_{22} = 50 with cost = 1260. Warehouse 1 supplies 10 units to retailer 2 and 90 units to retailer 3. Warehouse 2 supplies 80 units to retailer 1 and 50 units to retailer 2. Twenty units from the first warehouse are not supplied to anybody.

Based on Illustrations 2.1 and 2.2, we now address certain points in the forms of questions and answers:

1. Should we define X_{ij} to be integers?

No. For the above formulation of the transportation problem, It is not necessary to define X_{ij} as integers. It is enough to define them as continuous variables. It is observed that the LP relaxation of the transportation problem gives integer valued solutions. This will happen if the individual supply and demands are integers. This is due to the unimodularity property of the matrix of coefficients of the constraints. If the supply and demand quantities are non integers, then X_{ij} can also be non integers.

2. In this example, the total supply exceeded the total demand. If the supplies were 100 and 120 what happens to the solution?

We solve the example with supplies as 100 and 120. The solution terminates with a message **"Solver could not find a feasible solution".** It gives us the solution shown in Table 2.10 where we observe that one of the constraints is violated.

Table 2.10 – Solution when the supplies are 100 and 120.

	x11	x12	x13				
	x21	x22	x23				
	80	0	20		4	6	5
	0	50	70		5	7	8
obj fn	1330						
Cons	100	<=	100				
	120	<=	120				
	80	>=	80				
	50	>=	60				

	90	>=	90				

In this case the total supply is less than the total demand. The problem is therefore infeasible. From Table 2.8 we observe that we are unable to meet the demand of 60 of the second retailer. Whenever the total supply is less than the total demand, our formulation will give an "infeasible solution" and the solver will terminate with the message *"Solver could not find a feasible solution".*

3. Can we modify the problem by saying that we wish to transport the available quantities to meet the demand as much as possible (when the total supply is less than total demand) and minimize cost?

Yes, we can solve the problem by assuming that we want to meet as much of the demand as possible. We create an imaginary supply with 10 units (which is the difference between the total demand and total supply). With this imaginary supply we can meet the demand of all the retailers. We now have 3 warehouses and 3 retailers. We solve the problem using the excel solver and get the result shown in Table 2.11

Table 2.11 – Result from solver

	x11	x12	x13				
	x21	x22	x23				
	x31	x32	x33		4	6	5
	0	10	90		5	7	8
	80	40	0				
	0	10	0				
obj fn	1190						
cons	100	<=	100				
	120	<=	120				
	10	<=	10				
	80	>=	80				
	60	>=	60				
	90	>=	90				

The solution is $X_{12} = 10$, $X_{13} = 90$, $X_{21} = 80$, $X_{22} = 40$ and $X_{32} = 10$ with cost = 1190. Since warehouse 3 is a nonexistent supply, we now assume that retailer 2 gets only from warehouses 1 and 2 and gets 50 instead of 60. The demand is not fully met. The cost is 1190.

In this formulation we defined the cost of transportation from the imaginary warehouse to the retailers to be zero. The imaginary supply created to ensure that we are able to transport, is called a **dummy supply**.

4. Should we verify whether total demand is more than total supply whenever we solve a transportation problem? Otherwise we may get an infeasible situation.

Yes. The first step is to check if total supply is more than the total demand. *If it is less, we add an imaginary supply and balance the supply and demand.* Otherwise the standard LP formulation will give an infeasible solution. A transportation problem where total supply and total demand are equal is called a **balanced transportation problem**.

It is not necessary to balance it when total supply exceeds total demand, when we are solving the basic transportation problem as a LP through a solver.

(In the case of total demand exceeding total supply, It is also possible to modify the formulation such that we are able to transport all the available supplies at minimum cost to meet as much demand as we can. This is shown in illustration XX. In this case we need not add a dummy supply).

5. In the formulation of any balanced transportation problem, can we replace all the inequalities with equations? Is it a good thing to do?

In a balanced transportation problem, the total supply quantity is equal to the total demand. Therefore all the supply points will send all the quantities to the demand points (including the additional supply or demand point created to balance the problem) and all the demand points will receive exactly the demand. When we are solving small sized problems using a solver, it does not matter whether we use inequalities or equations.

If we are solving the transportation problem using special algorithms that are computationally faster, we use equations instead of inequalities. These algorithms solve only balanced problems.

6. In Illustration 2.2, we observe that warehouse 1 sends to two retailers and warehouse 2 also sends to two retailers. Will the warehouses always send to equal number of retailers? There are four movements in the solution. Is there a limit on these movements? Also retailer 2 is supplied by both the warehouses. Is it necessary that some demand points will be met by more than one supply point and vice versa?

In a balanced transportation problem with *m* supply points and *n* demand points we will have *m+n-1* allocations. This is because the LP would have *m+n* constraints and therefore is expected to have *m+n* basic variables in the solution. Since the total supply is equal to the total demand, we have a linearly dependent system of equations as the constraints.

Therefore there are only m+n-1 linearly independent constraints variables and therefore there will be *m+n-1* variables in the solution. The transportation problem is a degenerate LP.

When we have less than m+n-1 variables in the optimum solution to a balanced TP, it indicates that the transportation problem itself is degenerate.

It is not absolutely necessary that each warehouse will supply to a fixed number of retailers nor is it necessary that each retailer will get from a fixed number of warehouses.

7. Should we solve the transportation problem as an LP problem? Since we are minimizing the total cost, will an algorithm that progressively allocates based on minimum cost be optimal?

It is possible to create a solution where we progressively allocate based on minimum cost.

Consider Illustration 2.2. We can find the variable with minimum cost, X_{11} and allocate 80 to it. The next minimum cost is for X_{13} and we allocate 20. We have to allocate X_{23} = 70 and X_{22} = 60 with a total cost of **1400**. This gives us a solution with higher cost than 1260.

This method is called the **Minimum cost method**. Though it provides us with a good solution with very little computing effort, it does not guarantee the optimal solution.

We could think of another method that is intuitive in nature. Ideally we would allot maximum quantity to a variable with minimum cost. If we are unable to do so, we incur a penalty which is the difference between the lowest cost and the next lowest cost. We calculate the penalties for the 2 rows and 3 columns (in table 4.1). The penalties are 1 and 2 for the rows and 1, 1 and 3 for the columns. The maximum penalty is for column 3 and we therefore allot maximum possible to the minimum cost in this column. X_{13} = 90. The penalties are calculated for the two rows and for columns 1 and 2. They are 2, 2 and 1, 1 respectively. We choose X_{11} and allot 10 to it. The other allocations are X_{21} = 70 and X_{22} = 60 with cost = 1260 which is optimal.

The above method is called **penalty cost method** or **Vogel's approximation method** (Reinfield and Vogel, 1958) and gives very good solutions to the basic transportation problem. However, this method also does not guarantee optimum solution always.

8. Can we get the optimum solution from the solution using the Minimum cost method?

Yes. Consider the solution obtained using the minimum cost method: X_{11} =80, X_{13} =20, X_{23} = 70 and X_{22} = 60 with a total cost of 1400. The unallocated positions (variables) are X_{12} and

X_{21}. Allocating a maximum of 70 to X_{21} would reallocate the rest of the allocations and would result in the solution X_{13} = 90. X_{21} = 70, X_{22} = 60 and X_{11} = 10 with cost = 1260.

Now the unallocated positions are X_{12} and X_{23}. Allocating to these does not bring down the total cost and hence the above solution is optimal.

This method is called the Stepping stone method (Charnes and Cooper, 1954) and is used to get the optimum solution from the Minimum cost method solution or VAM solution for small sized problems.

9. Should we solve the transportation problem using the LP solvers or it is better to solve those using special algorithms?

If we have access to LP solvers, it is always possible to solve the transportation problem using the solver as an LP problem. In addition, researchers have developed special algorithms using starting solutions such as Vogel's approximation method and then optimizing those using methods such as Stepping Stone method or MODI method. These can be used to solve the transportation problem faster than solving as an LP problem. Some solvers have these algorithms available built in to solve the transportation problem. These algorithms are usually taught in the classroom in a course in Operations Research. These algorithms overcome the degenerate nature of the LP when applied to balanced problems.

Unbalanced transportation problems are those where the total supply exceeds the total demand or when the total demand exceeds the total supply.

When the total supply exceeds total demand, we can use the formulation with inequalities. It will transport quantities such that the demands are met and some items will remain in some of the supply points.

When the total demands exceed total supply, the formulation with inequalities will lead to an infeasible solution because we do not have the quantities to meet the demands. It is customary to find out how the available supplies are transported and to find out which demand points get less than the demand and by how much. We add a nonexistent supply point (dummy) whose supply is enough to balance the total demand. The cost of transporting from the dummy source to the all destinations is taken as zero. With the inclusion of the dummy supply the problem becomes balanced and the constraints can be taken as equations. The transportation algorithm can also be used because it solves a balanced transportation problem.

It is also customary to solve the case where the total supply exceeds the total demand. Here a dummy demand is created such that total supply and demand are balanced. The transportation from all the sources to the dummy demand is taken as zero. Again the transportation algorithm can be used to solve the balanced transportation problem.

In this course we assume that the reader can solve transportation problem using either LP or the transportation algorithm. In few illustrations we show the solver output for easy understanding. In subsequent illustrations we show the formulations and go directly to the solutions.

Maximization objective and use of big M

The transportation problem is a minimization problem. We can have a maximization version where we maximize the total profit where C_{ij} is the profit obtained by transporting a unit from i to j.

Sometime we cannot transport between specific supply points and demand points. It is customary to represent the cost there using a big M (large and positive) so that the large value of cost would prevent allocation of transportation quantities there.

Illustration 2.3

A company has four factories where TV cars are made and these are sold in three retail outlets. The cost of making the car in factory i is 4, 4.5, 3.8 and 4.2. The cost of transporting the car from factory i to retailer j is given in Table 2.12. The supplies in the warehouses are 200, 160, 180 and 150. The demands are 210, 230 and 200. The cars are sold at 6, 7 and 6.5 in the three retail centres. It is not possible to transport from factory 3 to retailer 1 due to security reasons. Find the transportation quantities that maximize the total profit? What happens to the solution if the demand increases by 50 in all the retailers?

Table 2.12 – Unit cost of transportation

	R1	R2	R3
F1	0.3	0.4	0.2
F2	0.15	0.25	0.22
F3	xxx	0.16	0.19
F4	0.07	0.09	0.1

The profit associated with selling a car produced in factory i in retailer j $P_{ij} = R_j - S_i - C_{ij}$ where R_j is the selling price at retailer j, S_i is the cost of production at factory i and C_{ij} is the cost of transporting one unit from i to j

$P_{11} = R_1 - S_1 - C_{11} = 6 - 4 - 0.3 = 1.7$

$P_{12} = R_1 - S_2 - C_{12} = 7 - 4 - 0.4 = 2.6$

The profit matrix is given in Table 2.13. The supplies and demands are also shown in the Table. We solve a maximization problem to maximize the total profit.

Table 2.13 – Profit matrix

	R1	R2	R3	
F1	1.7	2.6	2.3	200
F2	1.35	2.26	1.78	160
F3	-1000	3.04	2.51	180
F4	1.73	2.71	2.2	150
	210	230	200	

The problem is unbalanced with total supply = 690 exceeding the total demand = 640. We can formulate the balanced problem with inequalities. We have used unit profit = -1000 (equivalent of – big M for a maximization problem). We can either leave out the variable X_{31} on our formulation or include it with cost = -1000 so that the large negative cost leaves the variable out of the solution (maximization). If we use the special transportation algorithms that solve balanced problems we have to include variable X_{31} and include a large negative cost.

The formulation is to

Maximize $1.7X_{11} + 2.6X_{12} + 2.3X_{13} + 1.35X_{21} + 2.26X_{22} + 1.78X_{23} - 1000X_{31} + 3.04X_{32} + 2.51X_{33} + 1.73X_{41} + 2.71X_{42} + 2.2X_{43}$

Subject to

$X_{11} + X_{12} + X_{13} \leq 200$

$X_{21} + X_{22} + X_{23} \leq 160$

$X_{31} + X_{32} + X_{33} \leq 180$

$X_{41} + X_{42} + X_{43} \leq 150$

$X_{11} + X_{21} + X_{31} + X_{41} \geq 210$

$X_{12} + X_{22} + X_{32} + X_{42} \geq 230$

$X_{13} + X_{23} + X_{33} + X_{43} \geq 200$

$X_{ij} \geq 0$ and integer

The optimum solution to the maximization transportation problem is $X_{13} = 200$, $X_{21} = 160$, $X_{32} = 180$, $X_{41} = 50$, $X_{42} = 100$ with total profit = 1580.7. The given problem is not balanced since the total supply is 640 and the sum of the minimum demands is 590. From the solution we find that because it is a maximization problem, retailer 2 is allotted the extra 50 units (because of the ≥ constraint).

If the demands in the retailers are fixed and they do not take extra items, the demand constraints become equations. The optimum solution then is $X_{13} = 200$, $X_{21} = 160$, $X_{32} = 180$, $X_{41} = 50$, $X_{42} = 100$ with total profit = 1464.2. Supply point 2 has 50 units not allotted, which is the difference between the total supply and total demand.

If we change the demand values to 260, 280 and 250 in our formulation and solve the problem, we will get an infeasible solution. We have to create a dummy supply with capacity = 100. We assume that we can transport to all the demand points with zero cost. The formulation becomes

Maximize $1.7X_{11} + 2.6X_{12} + 2.3X_{13} + 1.35X_{21} + 2.26X_{22} + 1.78X_{23} - 1000X_{31} + 3.04X_{32} + 2.51X_{33} + 1.73X_{41} + 2.71X_{42} + 2.2X_{43}$

Subject to

$X_{11} + X_{12} + X_{13} \leq 200$

$X_{21} + X_{22} + X_{23} \leq 160$

$X_{31} + X_{32} + X_{33} \leq 180$

$X_{41} + X_{42} + X_{43} \leq 150$

$X_{51} + X_{52} + X_{53} \leq 100$

$X_{11} + X_{21} + X_{31} + X_{41} + X_{51} = 260$

$X_{12} + X_{22} + X_{32} + X_{42} + X_{52} = 280$

$X_{13} + X_{23} + X_{33} + X_{43} + X_{53} = 250$

$X_{ij} \geq 0$ and integer

The optimum solution to the problem using LP or the transportation algorithm (this problem is balanced) is given by $X_{13} = 200$, $X_{21} = 160$, $X_{32} = 180$, $X_{42} = 100$, $X_{43} = 50$, $X_{51} = 100$ with total profit = 1604.2. From the optimum solution we observe that demand point 1 is supplied by the dummy supply point. This demand point effectively does not get that supply of 100 and gets a total of 150.

Upper bounded transportation problem

The upper bounded transportation problem has upper bounds on the quantities that can be transported between the supply and demand points. The restriction comes due to availability of vehicles between chosen points. It could also come due to other considerations. The upper bounded transportation problems are also called capacitated transportation problems.

The formulation of the capacitated transportation problem is given by

Let X_{ij} be the quantity transported from warehouse (supply) i to destination (demand) j.

There are m supply points and n destinations.

The objective is to Minimize $\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij} X_{ij}$

Subject to

$$\sum_{j=1}^{n} X_{ij} \le a_i \ \forall i$$

$$\sum_{i=1}^{m} X_{ij} \ge b_j \ \forall j$$

$X_{ij} \le U_{ij}$

X_{ij} integer

This problem has mn more constraints compared to the transportation problem. The problem can be solved as an LP problem including the additional constraints. It can also be solved using special algorithms similar to Simplex method for bounded variables where the bounds are not explicitly treated as constraints.

The transportation problem with an upper limit on the quantity transported is called the **capacitated transportation problem** or **upper bounded transportation problem**. Here we add the constraint set $X_{ij} \le U_{ij}$ to the basic formulation. There are mn more constraints.

Illustration 2.4

Consider the data in Illustration 2.2. The demand in the three retailers is 80, 60 and 90 while 100 and 120 sets are available in the two warehouses. The costs of transportation are given in Table 2.14. If we have a restriction that we can transport a maximum of 60 (say) from a warehouse to a retailer, what happens to the solution?

Table 2.14 – Supply, demand and costs for a transportation problem

4	6	5	100
5	7	8	150
80	60	90	

We have to add constraints $X_{ij} \le 60$ for all i and j to our formulation. We solve the resultant formulation with 6 variables and 11 constraints to get the solution given in Table 2.15

Table 2.15– Solution given by the solver

	x11	x12	x13				
	x21	x22	x23				
					4	6	5
	40	0	60		5	7	8
	40	60	30				
	0	0	0				
obj fn	1320						

cons	100	<=	100		40	<=	60
	130	<=	150		0	<=	60
	80	>=	80		60	<=	60
	60	>=	60		40	<=	60
	90	>=	90		60	<=	60
					30	<=	60

The solution is $X_{11} = X_{21} = 40$; $X_{13} = X_{22} = 60$; $X_{33} = 30$ with cost = **1320.**

The cost associated with the optimum solution to this problem is higher than that of the the basic transportation problem and the *m+n-1* rule does not apply here.

We can modify the minimum cost method suitably. We allot $X_{11} = 60$, $X_{13} = 40$, $X_{21} = 20$, $X_{22} = 60$ and $X_{23} = 50$ to get a feasible solution with cost = 1360.

Modifications to the formulation to address different situations

We have earlier discussed a situation where the total supply was less than total demand and suggested that we create a dummy supply and solve a balanced problem. At the end, whatever quantity allocated from the dummy supply will not be actually available and therefore the corresponding demand will be short by that quantity. If we had not added the dummy and balanced the supply and demand, the solver would have resulted in an infeasible solution for the basic formulation.

We now modify the formulation to meet the situation where total supply is less than total demand and we wish to transport all the available supplies at minimum cost.

Illustration 2.5

Consider the problem of transporting TV sets of a particular brand (TVX) from 2 warehouses to 3 retail shops. The demand in the three retailers is 80, 60 and 90 while 100 and 120 sets are available in the two warehouses. The cost of transportation is defined as a unit cost of transporting an item from a warehouse to retailer and these are given in Table 2.16. Find the optimum solution with least total cost of transportation?

Table 2.16 – Supply, demand and costs for a transportation problem

4	6	5	100
5	7	8	120
80	60	90	

In this illustration we observe that total supply is less than total demand and we wish to supply all the available quantities at minimum cost. The formulation is as follows:

Minimize $4X_{11} + 6X_{12} + 5X_{13} + 5X_{21} + 7X_{22} + 8X_{23}$
Subject to
$X_{11} + X_{12} + X_{13} = 100$
$X_{21} + X_{22} + X_{23} = 120$
$X_{11} + X_{21} \leq 80$
$X_{12} + X_{22} \leq 60$
$X_{13} + X_{23} \leq 90$
$X_{ij} \geq 0$ and integer.

The solver solution is shown in Table 2.17

x11	x12	x13		4	6	5
x21	x22	x23		5	7	8
0	**10**	**90**				
80	**40**	**0**				
obj =	**1190**					
100	=	100				
120	=	120				
80	<=	80				
50	<=	60				
90	<=	90				

The quantities transported are $X_{12} = 10$, $X_{13} = 90$, $X_{21} = 80$, $X_{22} = 40$ with cost = 1190. Retailer 2 gets 10 units less because the total demand exceeds total supply by 10. The equations ensure that all the supplies have to be transported. The <= type constraints for the demand prevent the infeasibility from happening.

In situations where total supply exceeds total demand, the retailers would like to take the extra items. If more than one retailer claims extra items or if the total extra items demanded exceeds the available extra items, the formulation has to be suitably modified.

Illustration 2.6

Consider the data in Table 2.18. Assume that all the three retailers know that the total supply is 250 and total demand is 230. Retailer 2 is willing to take up to an additional 20 units while Retailer 3 is willing to take up to an additional 10 units. How much does each retailer get?

For the same data if Retailer 3 is willing to take any additional amount that is given while retailer 1 is willing to take a maximum of 10. Retailer 2 says that he will take either all 20 or does not want any additional amount. How much does each retailer get?

Table 2.18 – Transportation costs

4	3	5	100
5	7	8	150
80	60	90	

Retailer 2 is willing to take up to an additional 20 units while Retailer 3 is willing to take up to an additional 10 units. Two additional constraints

$X_{12} + X_{22} \leq 80$ and $X_{13} + X_{23} \leq 100$ are included. If we solve the problem with the addition of these two constraints, we would get the same solution $X_{12} = 10$, $X_{13} = 90$, $X_{21} = 80$ and $X_{22} = 50$ with cost = 1260 because it satisfies the two additional constraints.

The extra 20 units have to be transported to retailer 2 or 3 which means that all the quantities available with the 2 warehouses have to leave. The two supply constraints become equations. They become

$X_{11} + X_{12} + X_{13} = 100$ and $X_{21} + X_{22} + X_{23} = 150$. Retailer 1 should get exactly 80 and the excess 20 should not go to retailer 1. The constraint becomes an equation

$X_{11} + X_{21} = 80$. The formulation is now to

Minimize $4X_{11} + 6X_{12} + 5X_{13} + 5X_{21} + 7X_{22} + 8X_{23}$

Subject to

$X_{11} + X_{12} + X_{13} = 100$

$X_{21} + X_{22} + X_{23} = 150$

$X_{11} + X_{21} = 80$

$X_{12} + X_{22} \geq 60$

$X_{13} + X_{23} \geq 90$

$X_{12} + X_{22} \leq 80$

$X_{13} + X_{23} \leq 100$

$X_{ij} \geq 0$.

The solution is given in Table 2.19

Table 2.19 – Solution from solver

	x11	x12	x13				
	x21	x22	x23				
					4	6	5
	0	9.32E-11	100		5	7	8
	80	70	0				
obj fn	1390						
cons	100	=	100				
	150	=	150				
	80	>=	80				
	70	>=	60				
	100	>=	90				
	70	<=	80				
	100	<=	100				

The solution is X_{13} = 100; X_{21} = 80 and X_{22} = 70 with cost = **1390.** Retailers 2 and 3 get extra 10 units each and all the units available with the warehouses are distributed (at an additional cost) because the retailers are willing to take them.

Let X_{ij} the quantity transported from i to j to meet the committed demand (80, 60 and 90) respectively. Let Y_{ij} be the additional quantity sent to retailer j from supply i. The objective function is to

Minimize $\sum_{i=1}^{2}\sum_{j=1}^{3} C_{ij}X_{ij} + \sum_{i=1}^{2}\sum_{j=1}^{3} C_{ij}Y_{ij}$

Subject to

$$\sum_{j=1}^{n} X_{ij} + \sum_{j=1}^{n} Y_{ij} = a_i$$

The quantity of sets received by a retailer should meet the demand. This is given by

$$\sum_{i=1}^{m} X_{ij} = b_j$$

For the additional 20 items, Retailer 1 says that he would accept a maximum of 10 while retailer 3 can accept a maximum of 20. These constraints are

$$Y_{11} + Y_{21} \le 10$$

$Y_{13} + Y_{23} \leq 20$

Retailer 2 will accept 0 or 20. We define Z_2 = 1 if he gets 20 and 0 if he does not get anything. The constraint becomes

$Y_{12} + Y_{22} \leq 20Z_2$

The problem is formulated in the excel solver and solved. The optimum solution is given in Table 2.20

Table 2.20 – Optimum solution from solver

	x11	x12	x13	y11	y12	y13
z2	x21	x22	x23	y21	y22	y23
0	0	0	90	0	0	10
	80	60	0	10	0	0
obj fn	**1370**					
cons	100	=	100			
	150	=	150			
	80	=	80			
	60	=	60			
	90	=	90			
	10	<=	10			
	10	=	20			
	0	<=	0			

The optimum solution now allots extra 10 to Retailer 1 and an additional 10 to retailer 3. The total cost is 1370.

ASSIGNMENT PROBLEM AND ITS VARIANTS

Introduction to Lecture

S: Good morning.

T: Good morning. Welcome again. Today we will see the third exercise.

S: I think this one is going to be tougher than the previous exercise..

T: But you handled the previous exercise very well. Hope you enjoyed both the exercises.

S: Certainly yes. It is getting interesting.

T: Let us go directly to the question.

T: I am giving you 25 numbers, five in each row and 5 in each column. These are shown in Table 3.1

Table 3.1- Twenty five numbers

11	6	9	18	11
13	20	6	12	14
5	4	6	6	7
18	9	12	17	15
12	7	15	20	11

You have to choose exactly 5 numbers, one in each row and one in each column such that the sum of the five numbers is the smallest.

S: Looks interesting. First let me try and get an answer. Is 1 to 1, 2 to 2, 3 to 3, 4 to 4 and 5 to 5 correct? The sum is ***65****.*

T: This answer meets the condition that the five numbers are that there is one in each row and one in each column. The sum is 65 but is it the smallest sum?

S: I agree that the sum may not be the smallest. At least I got a solution that meets the conditions. Can I call this answer as 1-1, 2-2, 3-3, 4-4, 5-5?

T: Let us follow that representation. Can you do better than the answer that you gave?

S: Yes. I can consider 1-5, 2-4, 3-3, 4-2, 5-1 with sum = ***50****, which is better than the earlier total. I think I can get a better answer.*

S: I can pick the smallest number which is 3-2. This eliminates row 3 and column 2. The next smallest number is 6 which is 2-3. This eliminates row 2 and column 3. The next smallest number is 11 which is 1-1. This eliminates row 1 and column 1. The next smallest number is 11 (5-5). The only available number is 17, which is 4-4. The answer is 1-1, 2-3, 3-2, 4-4 and 5-5 with total = ***49*** *which is slightly better than 50.*

T: Does this give the smallest total?

S: I can't say. It should give the smallest total since I picked the smallest possible number at every stage.

T: You did. Did you realize that you had no choice regarding the last number?

S: Yes. Therefore I cannot say that this is the smallest total. Let me try solving the problem in a different way.

---

*S: The minimum values in the five columns are 5, 4, 6, 6, 7. I can choose 3-1, 1-2, 2-3. Out of the remaining numbers I can choose 4-4 and 5-5 with a total of **45**. It seems to be getting better and better.*

T: I have a few questions. You did not choose the minimum consistently. If you had considered the minimum, you should have chosen 3-2 and 2-3 which you did not.

S: I agree. I was carried away by a set of smaller numbers. Let me try and see whether this is the best answer or whether I can improve it.

T: Well. If this is not the best answer, how do you get it from this answer?

S: If this is not the best answer, then one new number should replace an existing number. The present solution is 3-1, 1-2, 2-3, 4-4 and 5-5. I can consider them in pairs and verify whether switching the positions can decrease the total. If I take 3-1 and 1-2, they can be replaced by 3-2 and 1-1. This increases the total. Starting with 3-1, I can try the remaining four pairs and do it for the rest of the numbers.

I am unable to decrease the total. It remains at 45. This should therefore be the minimum total possible.

T: Not necessarily. You tried only two at a time. You may try them three at a time and may get better answers. For example you can take 3-1, 1-2, 2-3 and replace with 3-3, 1-2, 2-1 or 3-3, 1-1, 2-2 or 3-2, 1-3, 2-1 or 3-2, 1-1, 2-3 and so on..

S: Looks like there are many possibilities. Though I have got a good answer, I can't say that it is the best.

T: Let me see if I can lead you to a different answer following a different approach. Let us start by finding the minimum in each row because you have to pick one from each row.

S: The values are 6, 6, 4, 9, 7. But some of them are in the same column. Therefore I cannot pick all of them. In a column I can choose one number only.

T: Therefore in some of the rows you have to choose the second minimum.

S: There is an increase in the total because of this.

T: Let us call this increase as the penalty of not being able to choose the minimum.

S: Is the penalty the difference between the smallest and the next bigger number?

T: Yes.

S: The five penalties for the rows are 5, 6, 1, 3, and 4. Row 2 has the highest penalty of 6.

T: You would pick 4 from row 3 but the increase of not choosing it is 1 (which is the penalty for row 3). If you do not choose 6 from row 2 you have to incur an extra 6.

S: Now I better choose 6 in row 2 first so that I can minimize the extra increase.

T: Can you now repeat this for the remaining four rows. The table with remaining four rows and four columns are given in Table 3.2

S: The penalties for rows 1, 3, 4 and 5 are 5, 1, 6, 4. I choose 9 for row 4 so that I can avoid the increase of 6.

T: Please continue.

S: The new penalties for rows 1, 3 and 5 are 0, 1 and 1. I choose 5 for row 3. The penalties for rows 1 and 5 are 7 and 9. I choose 11 for row 5 and I am left with 18 for row 1. The solution is 1-4, 2-3, 3-1, 4-2 and 5-5 with sum = 49. This is not a very good answer. I have a better answer.

T: This happened because you again did not have any control over the last chosen number. You can reduce the total by trying out the exchange idea.

S: I can exchange 1-4, 3-1 with 1-1 and 3-4. The answer is 1-1, 2-3, 3-4, 4-2 and 5-5 with total = ***43****. This looks great.*

T: Can you say that this is the best possible answer.

S: No. The reason is the same as you mentioned before. Any exchange cannot guarantee the best answer unless you exhaust all possibilities. This leads to a large number of calculations.

T: Can we look at a way to find the best answer?

S: Provided you show in the end that the answer we get is the best.

T: Let us try the following method:

T: Let us choose the minimum from each column. What are the values?

S: 5, 4, 6, 6, 7

T: We try to allot these columns to the rows. We can allot 3-1 or 3-2 or 2-3 or 3-3 or 3-4 or 3-5. We actually allot 3-2 and 2-3 because of fewer values. We have rows 1, 4 and 5 not allotted and columns 1, 4 and 5 not allotted. We increase the values of these columns by 1 so that some rows can pick them up.

S: The column values are 6, 4, 6, 7, 8. we have retained the old values for the allotted columns.

T: Row 1 needs a column. So we reduce all the values such that we reach the minimum of the column values. We reduce row 1 by 2 to get 9, 4, 7, 16, 9. We reduce row 4 by 5 to get 13, 4, 7, 12, 10 and row 5 becomes 9, 4, 12, 17, 18. Columns 1, 4, 5 are not allotted (shown in bold)

This is shown below in Table 3.2

Table 3.2–Reduced matrix

	6	*4*	*6*	***7***	***8***

Row 1	*9*	*4*	*7*	*16*	*9*
Row 4	*13*	*4*	*7*	*12*	*10*
Row 5	*9*	*4*	*12*	*17*	*8*

Out of unallotted columns 1, 4 and 5, row 5 has one value equal. We allot 5-5. Rows 1 and 4 cannot get the remaining columns. Row 1 gets column 2 and row 4 gets column 3. The allocations are 1-2, 4-3, 5-5.

S: You have increased the number of allotments to three. Can you increase it every time.

T: I may not be able to. More importantly I am having one new allocation every time. The values of unallotted columns 1 and 4 increases. The unallotted rows 2 and 3 now reduce their numbers. We get Table 3.3

Table 3.3 – Reduced matrix

	7	*4*	*6*	***8***	*8*
Row 2	*13*	*20*	*6*	*12*	*14*
Row 3	*4*	*4*	*6*	*6*	*7*

Row 3 gets column 4 because the number is less. Row 2 gets column 3. The allocations are 1-2, 2-3, 3-4 and 5-5.

S: Now we have 4 allotments.

T: The unallotted column 1 has an increased value of 8. Row 4 (unallotted) retains the values. This is shown in Table3.4

Table 3.4 – Modified matrix

	8	*4*	*6*	*8*	*8*
Row 4	*13*	*4*	*7*	*12*	*10*

Row 4 cannot get column 1. It takes column 2. Now column 1 increases its value. Row 1 is unallocated. These are shown in Table3.5

Table 3.5 – Modified matrix

	9	*4*	*6*	*8*	*8*
Row 1	*9*	*4*	*7*	*16*	*9*

Now row 1 gets column 1. The allocations are 1-1, 2-3, 3-4, 4-2, 5-5 with total = ***43****.*

S: Does this method always give the minimum total?

T: It ordinarily should. This method may fail in some cases. In spite of this, this is a good way to get the best wherever it works. The underlying idea is that every unallocated column has to increase its value so that some row will claim it. The idea of reallocation also helps in getting the best solution.

S: This is difficult to follow and understand.

T: I do agree. Though the method is simple, the fact that it always gives the smallest value is difficult to understand. There are more advanced ways to get to the minimum answer always.

S: Is this problem easy or difficult? Can we get the minimum sum always?

T: OK. You gave two or three answers. Can you tell me how many answers are possible?

S: I think a large number of solutions exist where we have one number from each row and one number from each column.

T: Exactly how many are possible?

S: The first number can be chosen in 5 ways because any one of them can be chosen in row 1. The second number can be chosen in 4 ways, the third in 3 ways and so on. We have 5 x 4 x 3 x 2 x 1 = 120 solutions.

T: You have 5! (factorial) solutions.

S: I remember reading about these in permutations and combinations.

T: You choose the best out of 120 possible solutions.

S: This should be difficult.

T: Actually not so in this case. It does not get tougher as the problem becomes bigger.. There are better methods that always give us the minimum sum.

S: You finally managed to get a complicated problem. Right in the beginning you told me that it won't be tough.

T: You handled it well. You did get a very good answer to the specific problem. I think the description part became a little involved. I am sorry about it. We will surely see an easier problem next.

S: Thank you.

Assignment and Allocation problems

The Linear Assignment Problem also called Assignment Problem is a special type of a Linear programming problem. This problem has extensive applications in allocation problems where different entities are allocated or assigned to other entities. The Assignment problem is stated as follows:

Given n tasks and n people the problem is to assign or allocate or match the tasks and people such that each task is assigned to exactly one person and each person to only one task. There is a cost of allocating task i to person j and the objective is to minimize the total cost of allocation. The tasks and persons can be two sets of entities with the same cardinality (same number of elements)

Given *n* jobs and *n* people the problem is to assign the jobs to people at minimum cost such that each person gets exactly one job and one job goes to exactly one person. The cost of assigning job i to person j is C_{ij}.

The formulation of the assignment problem

Let X_{ij} = 1 if task i is assigned to person j (= 0 otherwise). Let C_{ij} be the given cost of assigning i to j. The objective is to minimize the total cost of allocation which is to

Minimize $\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} X_{ij}$

Each task goes to exactly one person. This is given by

$$\sum_{j=1}^{n} X_{ij} = 1 \quad \forall i$$

Each person gets only one task. This is given by

$$\sum_{i=1}^{n} X_{ij} = 1 \quad \forall j$$

X_{ij} = 0, 1.

The assignment problem is therefore a binary Integer Programming (IP) problem where there are an equal number of tasks and people..

The formulation of the balanced assignment problem has n^2 binary decision variables and 2n linear constraints.

The Assignment problem solved using special algorithms such as the Hungarian algorithm (Kuhn, 1955) and the auction algorithm (Bertsikas, 1990). Here we solve the problem using LP and explain why it is an LP problem in spite of the binary restriction on the variables.

Illustration 3.1

Consider the 4 job 4 person assignment problem whose cost matrix is given in Table 3.6

Table 3.6 – Cost of assignment

7	11	8	6
5	4	8	7
10	11	9	12
16	14	15	18

Let X_{ij} = 1 if job i is assigned to person j. The objective is to

Minimize $\sum_{i=1}^{4}\sum_{j=1}^{4} C_{ij} X_{ij}$

Each person gets exactly one job

$$\sum_{i=1}^{4} X_{ij} = 1$$

Each job goes to exactly one person. This is given by

$$\sum_{j=1}^{m} X_{ij} = 1$$

$X_{ij} = 0,1$

The above problem was formulated as a binary problem and solved using the excel solver. The optimum solution is given in Table 3.7

Table 3.7 – Optimum solution from solver

x11	x12	x13	x14	x21	x22	x23	x24
x31	x32	x33	x34	x41	x42	x43	x44
0	0	0	1	1	0	0	0
0	0	1	0	0	1	0	0
7	11	8	6	5	4	8	7
10	11	9	12	16	14	15	18
objfn	34						
cons	1	=	1				
	1	=	1				
	1	=	1				
	1	=	1				
	1	=	1				
	1	=	1				
	1	=	1				
	1	=	1				

The optimum solution is $X_{14} = X_{21} = X_{33} = X_{42} = 1$ with cost = 34. Job 1 goes to person 4, job 2 to person 1, job 3 to person 3 and job 4 to person 2.

Illustration 3.2

The Government has four projects (I to IV) and four companies (A to D) have bid for the projects. All the companies have bid for all the projects and the cost of allotting project i to company j is (in crores of rupees) given in Table 3.8. Find the least cost assignment?

Table 3.8 – Cost of assignment

	A	B	C	D
Project 1	12	14	9	11
2	8	7	6	7
3	16	14	18	15
4	22	24	21	20

The formulation of the Assignment problem is to Minimize

$12X_{11} + 14X_{12} + 9X_{13} + 11X_{14} + 8X_{21} + 7X_{22} + 6X_{23} + 7X_{24} + 16X_{31} + 14X_{32} + 18X_{33} + 15X_{34} + 22X_{41} + 24X_{42} + 21X_{43} + 20X_{44}$

Subject to

$X_{11} + X_{12} + X_{13} + X_{14} = 1$

$X_{21} + X_{22} + X_{23} + X_{24} = 1$
$X_{31} + X_{32} + X_{33} + X_{34} = 1$
$X_{41} + X_{42} + X_{43} + X_{44} = 1$
$X_{11} + X_{21} + X_{31} + X_{41} = 1$
$X_{12} + X_{22} + X_{32} + X_{42} = 1$
$X_{13} + X_{23} + X_{33} + X_{43} = 1$
$X_{14} + X_{24} + X_{34} + X_{44} = 1$

$X_{ij} = 0,1$

The assignment problem is solved as a binary IP and the optimum solution is given by $X_{13} = X_{21} = X_{32} = X_{44} = 1$ with $Z = 51$.

We now solve the Assignment problem as an LP problem by relaxing the binary variables into continuous variables. This means that we remove the constraints $X_{ij} = 0, 1$ and replace them by $X_{ij} \geq 0$. The decision variables now represent the proportion of project j allotted to bidder i. We now assume in our formulation that parts of a project can be done by different people.

Solving the LP optimally we get the solution $X_{13} = X_{21} = X_{32} = X_{44} = 1$ with $Z = 51$.

We observe that the LP optimum gives us an integer solution, which is the same as the optimum solution given by the IP. This solution also indicates that even if we consider the possibility of allocating parts of projects to different bidders, it is economical to assign each of the projects completely to one bidder. The assignment problem when solved as a binary LP would always give optimum solutions where the variables take binary values. This is due to the *unimodularity* of the coefficient matrix. Hence assignment problem is treated as an LP problem and solved considering continuous variables. Though it is an LP, algorithms such as Hungarian algorithm (Kuhn, 1955) that are faster than the simplex algorithm are used to solve the problem. Modern day solvers also use the assignment algorithms instead of Simplex and solve the problem.

The assignment problem can also have a maximization objective if the purpose is to assign tasks to people to maximize profit. If a consulting company has four consultants who have to be assigned to four different projects and depending on the person, they can charge a fee, they would allot consultants to work on the projects such that maximum fee could be charged.

If the data shown in Table 3.8 represented the profits instead of the costs, the optimum solution to the maximization assignment problem would be $X_{11} = X_{24} = X_{33} = X_{42} = 1$ with $Z = 61$.(There are multiple or alternate optimum solutions and the solver gives only one of the multiple solutions, which is shown)

It is also to be noted that unless otherwise stated, the assignment problem is a minimization problem.

Assignment Problem – Unequal number of tasks and people.

We can have situations where the number of tasks may be more than the number of people or can be fewer than the number of people. In such cases, how do we formulate and solve the problem? If the number of tasks were fewer, do we allocate all the tasks? Which one among the people does not get a task? Would the LP solution still lead us to binary values? We address these questions and issues using an example.

Illustration 3.3

Consider the situation where there are three projects but four people bid for it. The costs are given in Table 3.9

Table 3.9 – Cost of assignment

	A	B	C	D
Project 1	12	14	9	11
2	8	7	6	7
3	22	24	21	20

Solve an assignment problem to find the allocation of tasks to people?

The problem is a minimization assignment problem where $X_{ij} = 1$ if task i is allotted to person j. There are three tasks and four people. There are 12 binary variables. Each task should go to exactly one person. The constraints are

$X_{11} + X_{12} + X_{13} + X_{14} = 1$

$X_{21} + X_{22} + X_{23} + X_{24} = 1$

$X_{31} + X_{32} + X_{33} + X_{34} = 1$

One person does not get a task. We do not know who the person is. We therefore write the four constraints as

$X_{11} + X_{21} + X_{31} \leq 1$

$X_{12} + X_{22} + X_{32} \leq 1$

$X_{13} + X_{23} + X_{33} \leq 1$

$X_{14} + X_{24} + X_{34} \leq 1$

$X_{ij} = 0, 1.$

The objective function is to minimize the total cost of allocation. This is to Minimize $12X_{11} + 14X_{12} + 9X_{13} + 11X_{14} + 8X_{21} + 7X_{22} + 6X_{23} + 7X_{24} + 22X_{31} + 24X_{32} + 21X_{33} + 20X_{34}$

The optimum solution to the IP as well as to the LP is given by $X_{13} = X_{22} = X_{34} = 1$ with Z = 36.

We observe that since $X_{11} = X_{21} = X_{31} = 0$, person 1does not get a task. The remaining three people are assigned tasks.

It is observed that the problem continues to give solutions where the variables take binary values when solved as LP.

When solved using special algorithms such as the Hungarian algorithm that are designed for square matrices (where the number of tasks and number of people are equal, it is customary to introduce a dummy (or nonexistent) task so that the number of tasks is equal to the number of people. The cost to carry out this task is the same for all the people (it can be any arbitrary constant but is the same for all people). This is usually taken as zero for all the tasks. The solution using the Hungarian algorithm would give the same solution given by the LP.

It is also possible to do the same here and solve the 4 x 4 problem as an LP. The number of variables and constraints increases but the solution is the same.

The same can be carried out if we have fewer people than tasks. If the data in Table 3.8 is solved for four tasks and the first three people, the optimum solution is $X_{13} = X_{21} = X_{32} = X_{44} = 1$ with $Z = 31$. Task 4 is not allotted to any person.