Finite Countable, Uncountable Sets :→

Sets → Collection of well defined distinct objects is called a set.

functions → Let X and Y be two non-empty sets A function $f: X \to Y$ is a rule which assigns a unique value $y \in Y$ for each $x \in X$.

X is called domain of f, Y is called co-domain of f and $f(x) = \{y \in Y ; y = f(x)$ for some $x \in X\}$ is called range of f

For example → $X = \{1, 2, 3\}$ $Y = \{a, b, c, d\}$

$f: X \to Y$ as $f(1) = a$, $f(2) = b$, $f(3) = c$

$D_f = X$, Co-domain $= Y$

$R_f = \{a, b, c\}$

One-One function → $f: X \to Y$ is called one-one if whenever $y \in Y$ has a preimage, then it is unique i.e.,

$f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$ then

$x_1 = x_2$.

Example- $f: \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$

$f(g(x)) = g(x+a)$ where $a \in \mathbb{R}$

if $g_1(x+a) = g_2(x+a)$

$g_1(x-a+a) = g_2(x-a+a)$

$g_1(x) = g_2(x)$

Non example → $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$f(x) = |x|$

Onto function → $f: X \to Y$ is called onto function if for each $y \in Y$, $\exists$ some $x \in X$ st $\boxed{f(x) = y}$

Example- $f: W \to N$

$f(x) = x+1$

Non example → $f: N \longrightarrow N$

$f(x) = x+1$

[$\because$ 0 is not a N.N]

NOTE → $f: N \to N$

$f(x) = x - 1$

$\cdot$ $y = f(x) = x - 1$

(Not a fxn)

① $f: X \to X$ $f(x) = x$ 1

onto + 1-1

Identity function → $f: X \to X$, $f(x) = x$ is called identity function

(One - One + onto)

Constant function → $f: X \to X$, $f(x) = c$, where c is any constant

$X = \{c, 2\}$ iff the set has only

1-1 → 1 element

onto → 1 element

Invertible - Both one-one and onto

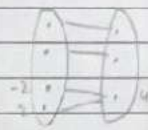
Points To Remember :→

$f: X \to Y$

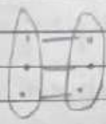
- ONLY ONE-ONE → $n(X) = n(f(x))$

$n(X) \leq n(Y)$

- If f is only onto → $n(Y) \leq n(X)$



- Bijective → $n(X) = n(Y)$



One-One Correspondence :→ If there exist a one-one onto mapping from set X to set Y we say that X and Y can be put in one-one correspondence OR we say that A and B have same cardinal number OR briefly X and Y are equivalent and we write it as $\boxed{X \sim Y}$.

Reflexive → $X \sim X$

$f : X \to X$

$f(x) = x$

Symmetric → $X \sim Y$ then $Y \sim X$

If $f : X \to Y$ is 1-1, onto

Define $g : Y \to X$

$g(y) = f^{-1}(y)$

Let $g(y_1) = g(y_2)$

$f^{-1}(y_1) = f^{-1}(y_2)$

$ff^{-1}(y_1) = ff^{-1}(y_2)$

$\boxed{y_1 = y_2}$ 1-1

$g(y) = f^{-1}(y)$ — Homework

Onto

$f : X \to Y$ is onto

$g : Y \to X$ is defined as $g(y) = f^{-1}(y)$

TP g is onto

Let $x \in X$

then $f(x) \in Y \Rightarrow f(x) = y$ for some $y \in Y$

$x = f^{-1}(y)$

$= g(y)$

Hence y is pre image of x under g

$\Rightarrow g$ is onto

Transitive → $X \sim Y$, $Y \sim Z$ then $X \sim Z$

$g \circ f : X \to Z$

PT → 1-1, onto

(i) f is 1-1

If $f(x_1) = f(x_2)$

then $x_1 = x_2$

g is 1-1

If $g(x_1) = g(x_2)$

then $x_1 = x_2$

We need to prove that,

If $gof(x_1) = gof(x_2)$ then,

$x_1 = x_2$

Suppose $gof(x_1) = gof(x_2)$

$g(f(x_1)) = g(f(x_2))$

Given g is 1-1

So $f(x_1) = f(x_2)$

And f is also 1-1

$x_1 = x_2$

Hence, if $gof(x_1) = gof(x_2)$ then $x_1 = x_2$

∴ gof is 1-1

Since $g : B \to C$ is onto

Suppose $z \in C$ then there exists a pre-image in B

Let the pre image be y

Hence $y \in B$ such that $g(y) = z$

Illy, since $f: A \to B$ is onto

If $y \in B$ then there exists a pre-image in A

Let the pre image be x

Hence, $x \in A$ st $f(x) = y$

Now, $g \circ f : A \to C$

$$g \circ f = g(f(x)) = g(y) = z$$

every x in A, there is an image z C

Thus $g \circ f$ is onto

set - If number of elements is finite.

set - ...

Infinite set

Countable sets   Uncountable sets

Countable Sets :- A set X is called countable if either it is finite or its cardinality (number of elements) is same as cardinality of Natural Numbers.

Uncountable Sets :- A set which is not countable is called uncountable sets

Examples - $N = \{1, 2, 3, ----\}$

$Z = \{----, -4, -3, -2, -1, 0, 1, 2, 3, 4 ---\}$

$N \subsetneq Z$ ⟶ strictly contains

① Define $f: N \to Z$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

f is well defined

TP - if $n_1 = n_2$
then $f(n_1) = f(n_2)$

If n_1 is even $\Rightarrow \frac{n_1}{2} = \frac{n_2}{2}$

$\Rightarrow f(n_1) = f(n_2)$

If n_1 is odd $\Rightarrow \frac{-(n_1-1)}{2} = \frac{-(n_2-1)}{2}$

$\Rightarrow f(n_1) = f(n_2)$

$\Rightarrow f$ is well defined.

<u>f is 1-1</u> Let $f(n_1) = f(n_2) > 0$

$\Rightarrow \frac{n_1}{2} = \frac{n_2}{2}$

$\Rightarrow n_1 = n_2$

$f(n_1) = f(n_2) \leq 0$

$\frac{-(n_1-1)}{2} = \frac{-(n_2-1)}{2}$

$n_1 - 1 = n_2 - 1$

$\Rightarrow n_1 = n_2$

$\therefore f$ is 1-1

<u>f is onto</u> Let $m \in \mathbb{Z}$

If $m > 0$ then $2m \in \mathbb{N}$ and $f(2m) = m$

If $m = 0$ then $f(1) = 0$

If $m \leq -1$ then $-m \geq 1$

$-2m \geq 2$

$1-2m \geq 3$

$\Rightarrow 1-2m \in \mathbb{N}$

$f(1-2m) = \frac{-(1-2m-1)}{2}$

$= +m$

$\therefore f$ is onto

<u>Sequence</u> → By a sequence, we mean a function f defined on the set of natural numbers.

$f : \mathbb{N} \longrightarrow \mathbb{R}$ (Domain - set of natural numbers)

If $f(n) = x_n$ then we denote the sequence f by symbol $\{x_n\}$.

<u>Theorem</u> → Every subset of a countable set A is countable.

Proof → Let $B \subseteq A$

If B is finite then clearly B is countable

If B is infinite

Arrange the elements of set A in a sequence where x_n denotes n^{th} element of A

Construct a sequence $\{n_K\}$ as follows :-

Let n_1 be the smallest positive integer st $x_{n_1} \in B$

Let n_2 be the smallest positive integer st $n_2 > n_1$ and $x_{n_2} \in B$

Having chosen $n_1, n_2, \dots, n_{k-1}$;

Let n_k be the smallest integer greater than n_{k-1} st $x_{n_k} \in B$

Defining $g : N \to B$ by $g(k) = x_{n_k}$ is 1-1 and onto
($\because f$ is 1-1, onto)

Hence B is countable.

[2] Theorem → The set $[0,1]$ is uncountable.

Proof → Suppose $[0,1]$ is countable.

$$[0,1] = \{x : 0 \le x \le 1\}$$

Each real number in $[0,1]$ has decimal representation $0 \cdot a_1 a_2 a_3$ where $a_{ij} \in N \ \forall i$

Since all real numbers in $[0,1]$ are countable

So, we can define a 1-1 correspondence of members of $[0,1]$ with set of natural no's in the following members.

$1 \leftrightarrow 0 \cdot a_{11} \, a_{12} \, a_{13} \dots$
$2 \leftrightarrow 0 \cdot a_{21} \, a_{22} \, a_{23} \dots$
$3 \leftrightarrow 0 \cdot a_{31} \, a_{32} \, a_{33} \dots$
and so on

Construct a number $x = 0 \cdot b_1 b_2 b_3 \dots$ in the following manner st

$$b_i \begin{cases} 4 & \text{if } a_{ii} = 5 \\ 5 & \text{if } a_{ii} = 4 \end{cases} \quad \forall i = 1, 2, 3, \dots$$

then $0 \le x \le 1$ and x is not in above mentioned list.

⇒ The set of real numbers in $[0,1]$ is not countable.

NOTE → $(-\infty, \infty)$ - uncountable
$[0,2]$ - uncountable

Imp Theorem → If $f : A \to B$ is 1-1 and B is countable, then A is also countable.

Proof → (i) If A is finite clearly A is countable

(ii) If A is infinite
Now $f(A) \subseteq B$

$\Rightarrow f(A)$ is countable

$A \xrightarrow{f} f(A) \xrightarrow{h} \mathbb{N}$

$hof : A \longrightarrow \mathbb{N}$ is 1-1 correspondence

$\Rightarrow A$ is countable

Theorem → Cartesian product of two countable sets is countable.

Proof → Let A, B be any two countable sets then,

their cartesian product is

$A \times B = \{(a, b) ; a \in A, b \in B\}$

If any one of A and B is empty, say A is empty, then

$A \times B$ is also empty so NTP

If any one of A and B is finite say A is finite.

$A = \{a_1, a_2, --- a_n\}$

$B = \{b_1, b_2, --- b_n\}$

$A \times B = \{(a_1, b_1) (a_1, b_2) - - -$

$(a_2, b_1) \; (a_2, b_2) - - -$

$\vdots$

$(a_n, b_1) \; (a_n, b_2) - - - \}$

Define $f : \mathbb{N} \longrightarrow A \times B$ by listing the elements like $(a_1, b_1) (a_2, b_1) - - - (a_n, b_1) (a_1, b_2) - - - (a_2, b_2) - - - (a_n, b_2) - - -$

which is 1-1 correspondence.

If both A and B are countable infinite, then we define a function by $f : A \times B \to \mathbb{N}$ by defining

$f(a_1, b_1) = 1$, $f(a_2, b_1) = 2$, $f(a_1, b_2) = 3$ and so on

① $f : \mathbb{Q} \longrightarrow \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$

$f\left(\frac{m_1}{n_1}\right) = f\left(\frac{m_2}{n_2}\right)$

$(m_1, n_1) = (m_2, n_2)$

$m_1 = m_2 \; , \; n_1 = n_2$

$\frac{m_1}{n_1} = \frac{m_2}{n_2}$

$\therefore$ It is 1-1 fxn.

Theorem :- Prove that the set of Rational Numbers is countable.

Proof :- We know that $\mathbb{Z}$ is a countable set (Already Proved)

also $\mathbb{Z} \setminus \{0\} \subseteq \mathbb{Z}$

$\Rightarrow \mathbb{Z} \setminus \{0\}$ is countable
(∵ subset of a countable set is countable)

$\Rightarrow \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable
(∵ cartesian product of two countable sets is countable)

Define $f : \mathbb{Q} \rightarrow \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ by

$f\left(\frac{m}{n}\right) = (m, n)$; where $\mathbb{Q} = \left\{\frac{m}{n} ;\ m, n \in \mathbb{Z},\ n \neq 0,\ (m,n) = 1\right\}$

f is 1-1 Let $\frac{m_1}{n_1}, \frac{m_2}{n_2} \in \mathbb{Q}$ st

$$f\left(\frac{m_1}{n_1}\right) = f\left(\frac{m_2}{n_2}\right)$$

$$(m_1, n_1) = (m_2, n_2)$$

$$m_1 = m_2 \ ; \ n_1 = n_2$$

$$\Rightarrow \frac{m_1}{n_1} = \frac{m_2}{n_2}$$

$\Rightarrow f$ is 1-1

$\Rightarrow \mathbb{Q}$ is countable (∵ If $f : A \rightarrow B$ is 1-1 fxn and B is countable, then A is also countable)

Theorem :- Let $\{A_i\}_{i \in \mathbb{N}}$ be a collection of countably many countable sets such that A_i's are mutually disjoint. Then $\bigcup_{i \in \mathbb{N}} A_i$ is also countable.

Proof :- Denote the elements of the set A_n by $\{a_{1n}, a_{2n}, \ldots\}$

Let $S = \bigcup_{n \in \mathbb{N}} A_n$

If $x \in S \Rightarrow x \in A_n$ for exactly one n.

$\Rightarrow x = a_{mn}$ for some $m \in \mathbb{N}$, $n \in \mathbb{N}$

Define $f : S \rightarrow \mathbb{N} \times \mathbb{N}$

$$f(x) = (m, n).$$

Let $f(x) = f(y)$
$(m_1, n_1) = (m_2, n_2)$
$m_1 = m_2 \ , \ n_1 = n_2$
$a_{m_1 n_1} = a_{m_2 n_2} \Rightarrow x = y$

$\Rightarrow f$ is 1-1

Hence S is countable (∵ if $f : A \rightarrow B$ is 1-1 fxn and B is countable, then A

Theorem - Let $\{A_i\}_{i\in\mathbb{N}}$ be a collection of countably many countable sets. Then, $\bigcup_{i\in\mathbb{N}} A_i$ is also countable

Proof: We define new sets in the following manner.

$B_1 = A_1$

$B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus (A_1 \cup A_2)$

and so on

$B_n = A_n \setminus (\bigcup_{i=1}^{n-1} A_i)$

then $\bigcup_{i\in\mathbb{N}} B_i = \bigcup_{i\in\mathbb{N}} A_i$

and LHS is countable ($\because$ B_i's are mutually disjoint)

$\Rightarrow$ RHS is also countable.

Theorem - set of rational numbers is countable.

Proof: for $n = 1, 2, 3, ---$

Let $A_n = \{\frac{m}{n} ; m \in \mathbb{N}$ and n is fixed$\}$

then $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{Q}^+$

$\Rightarrow$ $\mathbb{Q}^+$ is countable as each A_n is countable

Hly $\mathbb{Q}^-$ is countable

$\Rightarrow \mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$

$\Rightarrow \mathbb{Q}$ is countable

Theorem - The set of irrational numbers is uncountable

set of irrational $= \mathbb{R}\setminus\mathbb{Q} = \mathbb{R} \cap (\mathbb{Q})^c$

$\mathbb{Q} \cup (\mathbb{R}\setminus\mathbb{Q}) = \mathbb{R} \in (0,1)$

$\mathbb{Q}$ ↓ countable, $\mathbb{R}\setminus\mathbb{Q}$ ↓ uncountable

$\Rightarrow \therefore \mathbb{Q}^c$ is uncountable.

14/10/24 Metric Space :- Let X be a non-empty set. A metric d on X is a function $f: X\times X \longrightarrow \mathbb{R}$ such that

1. $d(x,y) \geq 0 \quad \forall\ x,y \in \mathbb{R}$
2. $d(x,y) = d(y,x) \quad \forall\ x,y \in \mathbb{R}$
3. $d(x,y) = 0$ iff $x = y$
4. $d(x,y) \leq d(x,z) + d(z,y) \ \forall\ x,y,z \in \mathbb{R}$

and (X,d) is a metric space.

Examples: 1. $X = \mathbb{R}$, Define $d: \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$

by $d(x,y) = |x-y|$.

Sol: (i) $d(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}$
$|x-y| \geq 0$
$x-y \geq 0$

(ii) $d(x,y) = |x-y|$
$d(y,x) = |y-x| = d(x,y)$

(iii) $d(x,y) = 0$
$\Leftrightarrow |x-y| = 0$
$\Leftrightarrow x-y = 0$
$\Leftrightarrow x = y$

(iv) $d(x,y) \leq d(x,z) + d(z,y)$

$|(x-z) + (z-y)|$
$\leq |x-z| + |z-y|$
$(\because |a+b| \leq |a| + |b|)$

$= d(x,z) + d(z,y)$.

$(\mathbb{R}, d)$ is a metric space

Let $X = \mathbb{R}$, Define $d: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$d(x,y) = |2x - 2y|$

) $d(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}$.
$2|x-y| \geq 0$
$x-y \geq 0$

$d(x,y) = 2|x-y|$

$d(y,x) = 2|y-x| = d(x,y)$

(iii) $d(x,y) = 0$
$\Leftrightarrow 2|x-y| = 0$
$\Leftrightarrow |x-y| = 0$
$\Leftrightarrow x = y$

(iv) $d(x,y) = 2|x-y|$
$= 2|(x-z) + (z-y)|$
$\leq 2|x-z| + 2|z-y|$

$d(x,y) \leq d(x,z) + d(z,y)$

3. Let $X = \mathbb{R}$, define $d: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$d(x,y) = (x-y)^2$

Sol: (i) $d(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}$
$(x-y)^2 \geq 0$
$x-y \geq 0$

(ii) $d(x,y) = (x-y)^2$
$d(y,x) = (y-x)^2 = d(x,y)$

(iii) $d(x,y) = 0$
$\Leftrightarrow (x-y)^2 = 0$
$\Leftrightarrow x-y = 0$
$\Leftrightarrow x = y$

(iv) $(x-y)^2 \leq (x-z)^2 + (z-y)^2$
ie, $x^2+y^2-2xy \leq x^2+z^2-2xz+z^2+y^2-2zy$

i.e., $2xz + 2yz - 2xy \leq 2$

$z = 0, \ x = -1, \ y = 1$

$\therefore$ It is not a metric space.

4. X - Arbitrary non-empty set

$d : X \times X \to \mathbb{R}$

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(Discrete metric space).

Sol: (i) $d(x,y) \geq 0$ is obvious

(ii) $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

$d(y,x) = \begin{cases} 1 & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases}$

(iii) $d(x,y) = 0$

If $x = y$ $\quad 0 = 0$

(iv) TP- $d(x,y) \leq d(x,z) + d(z,y)$

Case 1- $d(x,z) = 0 = d(z,y)$

$x = z, \ z = y \Rightarrow x = y$

$d(x,y) = 0$

$d(x,y) \leq d(x,z) + d(z,y)$

Case 2- $d(x,z) = 1 = d(z,y)$

$\Rightarrow x \neq z, \ z \neq y \Rightarrow x = y$ or $x \neq y$

$\Rightarrow d(x,y) = 0$ or 1

$\Rightarrow d(x,y) \neq d(z,y)$

$1 + 1 = 2$

Case 3- $d(x,z) = 0, \ d(z,y) = 1$

$x = z, \ z \neq y \Rightarrow x \neq y$

$d(x,y) = 1$

$d(x,z) + d(z,y)$

$0 + 1 = 1$

Case 4- $d(x,z) = 1, \ d(z,y) = 0$

$\Rightarrow x \neq z, \ z = y \Rightarrow x \neq y$

$d(z,y) = 1$

$d(x,z) + d(z,y)$

$1 + 0 = 1$.

Neighbourhood of a point :- Let (X,d) be a metric space. A neighbourhood of a point $p \in X$ is the set $\{q : d(p,q) < r\}$, $r > 0$ and r is called radius of neighbourhood.

It is denoted by $N_r(p)$.

NOTE - 1. On real line with usual metric

$N_r(p) = (p-r, p+r)$

2. On $\mathbb{R}^2$, Euclidean space,

$N_r(p)$ forms an opened disc with centre p and radius r.

3. In the discrete space,

$$N_r(p) = \begin{cases} \{p\} & \text{if } 0 < r < 1 \\ X & \text{if } r > 1 \end{cases}$$

$N_r(p) = \{q ; d(p,q) < r < 1\}$

$= \{q : d(p,q) = 1, d(p,q) = 0\}$

$= X$

$N_r(p) = \{q ; d(p,q) < r = 0\}$

$= \{q ; d(p,q) = 0\}$

$= \{q ; p = q\}$

$= \{p\}$

$\mathbb{C}$, $d(z_1, z_2) = |z_1 - z_2|$.

$d(z_1, z_2) \geq 0$

$|z_1 - z_2| \geq 0$

$z_1 - z_2 \geq 0$

(ii) $d(z_1, z_2) = |z_1 - z_2|$

$d(z_2, z_1) = |z_2 - z_1| = d(z_1, z_2)$

(iii) $d(z_1, z_2) = 0$

$\Leftrightarrow |z_1 - z_2| = 0$

$\Leftrightarrow z_1 - z_2 = 0$

$\Leftrightarrow z_1 = z_2$

(iv) $d(z_1, z_2) = |z_1 - z_2|$

$= |z_1 - z_3 + z_3 - z_2|$

$\leq |z_1 - z_3| + |z_3 - z_2|$

$= d(z_1, z_3) + d(z_3, z_2)$

$\therefore X = \mathbb{C}$ is a metric space.

18/10/21 **Limit Point** :- A point x is called the limit point of the set $E \subseteq X$ if every neighbourhood of p contains a point $q \neq p$ st $q \in E$ or $N_r(p) \setminus \{p\} \cap E \neq \phi$.

for example - if $E = (1,2) \subseteq \mathbb{R}$ then every point of E is a limit point of E and 1, 2 are also limit points of E. No other element of $\mathbb{R}$ is limit point of E.

Set of limit pts of $E = [1,2]$

Isolated point - If $p \in E$ and p is not limit point of E then p is called isolated point.

Closed set - E is called closed set if every limit point of E is a point of E.

$E = \{1, 3\}$ $\quad E' = \phi$ $\quad E' \subseteq E$

Interior point - A point p is interior of E if there is a neighbourhood $N(p)$ such that N is contained inside E i.e, $N \subseteq E$.

It is denoted by E°

NOTE: Every interior point is a limit point but converse is not true.

1) $E = (1,2) \cup \{3\}$

$E^{\circ} = (1,2)$ $\quad E' = [1,2]$

2) $E = [1,2]$

$E^{\circ} = (1,2)$ $\quad E' = [1,2]$

$E = \{1,3\}$ $\quad E^{\circ} = \phi$, $E' = \phi$

Open set - A set E is called open set if every point of E is an interior point of E.

Examples - $(1,2)$

Perfect set - A set E is called perfect if E is closed and every point of E is a limit point of E.

Example - $[1,2]$

Non example - $E = [1,2] \cup \{3\}$

$E' = [1,2]$ This is closed but not perfect

Dense set - A set E is called dense in X if every point of X is a limit point of E or a point of E.

Denoted by $\bar{E}$; $\boxed{\bar{E} = E \cup E' = X}$

Example - E - set of all rational numbers then $\bar{E} = X$

Theorem : Every neighbourhood is an open set.

Proof: Let X be given space and $p \in E$ be arbitrary element.

$N_r(p) = \{q \in X ; d(p,q) < r\}$

Let $q \in N_r(p)$

$\Rightarrow d(p,q) < r$

$\Rightarrow \exists$ a +ve real number 'h' such that

$d(p,q) = r-h$

We need to find some $\delta > 0$ such that

$N_\delta(q) \subseteq N_r(p)$

Let $\delta = h$ then if $x \in N_h(q)$ then

$d(x,q) < h$ and

$d(x,p) \leq d(x,q) + d(q,p)$
$< h + r - h$
$= r$

$\Rightarrow x \in N_r(p)$

10/2| Theorem → If p is the limit point of set E then every neighbourhood of p contains infinitely many points of E.

Proof: Suppose $\exists$ nbh $N_r(p)$ which contains finitely many points of E.

$\Rightarrow N_r(p) \setminus \{p\} \cap E = \{p_1, p_2, \ldots\ p_n\}$

Let $\delta = \min \{d(p, p_i),\ i = 1, 2, \ldots\ n\}$
then $\delta > 0$

Claim → $N_\delta(p) \setminus \{p\} \cap E = \phi$

Let $x \in N_\delta(p) \setminus \{p\} \cap E$
$\Rightarrow x \in N_\delta(p),\ x \neq p,\ x \in E$

$\Rightarrow d(p,x) < \delta,\ x \neq p,\ x \in E$
also $\delta < r$

$\Rightarrow x \in N_r(p) \setminus \{p\},\ x \in E$
$\Rightarrow x \in p_i$ for some $i = 1, 2, \ldots\ n$
$\Rightarrow d(x,p) \geq \delta$

which is a contradiction.

Corollary: A finite set has no limit point.

Proof: Let p is a limit point of finite set E, then each nbh of p must have infinitely many points of E which is not possible as E is finite.

$\Rightarrow$ E has no limit point

for example → $\boxed{S = \{1, 3\}}$
finite set
$\Rightarrow$ has no limit point.

Theorem: A set E is open iff E^c is closed.

Proof: Suppose E^c is closed

To prove: E is open.

Let $x \in E$ be any point

Claim: $\exists$ some nbh of x which completely lies in E.

if $x \in E \Rightarrow x \notin E^c$

$\Rightarrow x$ is not a limit point of E^c

$\exists$ some nbh of x st,

$$N_r(x) \setminus \{x\} \cap E^c = \phi$$
$$\Rightarrow N_r(x) \cap E^c = \phi$$
$$N_r(x) \subseteq (E^c)^c$$
$$= E$$

$\Rightarrow x$ is interior point of E

$\Rightarrow E$ is open.

Converse: Now, let E is open.

To prove: E^c is closed.

Let x be a limit point of E^c

TP: $x \in E^c$

$\Rightarrow N_r(x) \cap E^c \neq \phi$ for each nbh of x

$\Rightarrow N_r(x) \not\subseteq E$

$\Rightarrow x$ is not an interior point

$\Rightarrow x \notin E$

$\Rightarrow x \in E^c$

Corollary: A set E is closed iff E^c is open.

Proof: Let E^c is open

$\Rightarrow (E^c)^c$ is closed

$\Rightarrow E$ is closed

If E is closed

$\Rightarrow (E^c)^c$ is closed

$\Rightarrow E^c$ is open

Theorem: ① for any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open

② for any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

③ for a finite collection $G_1, G_2, \ldots, G_n$ of open sets, $\bigcap_{i=1}^{n} G_i$ is open.

④ for a finite collection $F_1, F_2, \ldots, F_n$ of closed sets $\bigcup_{i=1}^{n} F_i$ is closed.

Proof: ① Let $x \in \bigcup_\alpha G_\alpha$

$\Rightarrow x \in G_\beta$ for some β

but G_β is open

$\Rightarrow \exists N_r(x)$ such that

$N_r(x) \subseteq G_\beta \subseteq \bigcup_\alpha G_\alpha$

$\Rightarrow x$ is interior point of $\bigcup_\alpha G_\alpha$

$\Rightarrow \bigcup_\alpha G_\alpha$ is open.

(2) Given that F_α is closed $\forall \alpha$

$\Rightarrow F_\alpha^c$ is open $\forall \alpha$

$\Rightarrow \bigcup_\alpha F_\alpha^c$ is open

$\Rightarrow (\bigcap_\alpha F_\alpha)^c$ is open

$\bigcap_\alpha F_\alpha$ is closed.

(3) Let $x \in \bigcap_{i=1}^n G_i$

$\Rightarrow x \in G_i \;\; \forall \; 1 \le i \le n$

So, $\exists N_{r_i}(x)$ such that $N_{r_i}(x) \subseteq G_i$ for each i

Let $r = \min\{r_1, r_2 \text{------} r_n\}$

Now, consider $N_r(x)$

$N_r(x) \subseteq N_{r_i}(x) \;\; \forall i$

$N_r(x) \subseteq G_i$

$N_r(x) \subseteq \bigcap_{i=1}^n G_i$

$\Rightarrow x$ is an interior point of $\bigcap_{i=1}^n G_i$

So, $\bigcap_{i=1}^n G_i$ is open.

(4) Let F_i, $1 \le i \le n$ be closed sets

$\Rightarrow F_i^c$, $1 \le i \le n$ is open

$\bigcap_{i=1}^n F_i^c$ is open

$(\bigcup_{i=1}^n F_i)^c$ is open

$\Rightarrow \bigcup_{i=1}^n F_i$ is closed.

Examples:- (1) $\bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, a+2\right]$

$(a, a+2]$

(2) $\bigcap_{n=1}^\infty \left(\frac{-1}{n}, \frac{1}{n}\right)$ $\{0\}$

Sol^n (1) $\bigcup_{n \in \mathbb{N}} \left[a+\frac{1}{n}, a+2\right]$

$= [a+1, a+2] \cup \left[a+\frac{1}{2}, a+2\right] \cup \left[a+\frac{1}{3}, a+3\right]$

$\cup \left[a+\frac{1}{4}, a+2\right] \cup \text{------}$

$= (a, a+2]$.

(2) $\bigcap_{i=1}^\infty \left(\frac{-1}{n}, \frac{1}{n}\right)$

$= (-1, 1) \cap \left(\frac{-1}{2}, \frac{1}{2}\right) \cap \left(\frac{-1}{3}, \frac{1}{3}\right) \cap \left(\frac{-1}{4}, \frac{1}{4}\right) \cap \text{----}$

$= \{0\}$

Date 23/10/21

NOTE: finite set = No limit point = Closed

Theorem: Let (X, d) be a metric space and $E \subseteq X$ then.

1. $\bar{E}$ is closed
2. $E = \bar{E}$ iff E is closed
3. If $F \subset X$ is closed set $E \subset F$, then $\bar{E} \subset F$.

Proof: $\bar{E} = E \cup E'$

To prove that $\bar{E}$ is closed, we will prove that $\bar{E}^c$ is open.

Let $x \in \bar{E}^c \Rightarrow x \notin \bar{E} = E \cup E'$

$x \notin E$ and $x \notin E'$

$\Rightarrow x$ is not a limit point of E

$\Rightarrow \exists N_r(x), r > 0$ st

$N_r(x) \setminus \{x\} \cap E = \phi$

$\Rightarrow N_r(x) \cap E = \phi$

$N_r(x) \cap \bar{E} = \phi$

$\because N_r(x) \cap \bar{E} = N_r(x) \cap (E \cup E')$
$= (N_r(x) \cap E) \cup (N_r(x) \cap E')$
$= \phi \cup (N_r(x) \cap E')$
$= N_r(x) \cap E'$

Now if $N_r(x) \cap E' \neq \phi$ then

$\exists y$ st $y \in N_r(x) \Rightarrow y \in E'$

$\Rightarrow y \in N_r(x)$ and y is limit point of E

$\Rightarrow y \in N_r(x)$ and each neighbourhood of $y \setminus \{y\} \cap E \neq \phi$

$\Rightarrow x \in N_r(y)$ and $N_r(y) \setminus \{y\} \cap E \neq \phi$

$\Rightarrow x \in N_r(y) \cap E$

which is contradiction

$\Rightarrow N_r(x) \subseteq \bar{E}^c$

$\bar{E}^c$ is open

$\Rightarrow \bar{E}$ is closed.

(2) $E = \bar{E}$ then as $\bar{E}$ is closed

Clearly, E is closed.

Conversely: $\bar{E} = E \cup E' \subseteq E \cup E$
$= E$.

$\Rightarrow \bar{E} \subseteq E$

also $E \subseteq E \cup E'$
$= \bar{E}$

$\Rightarrow E \subseteq \bar{E}$

hence $E = \bar{E}$

(3) F is closed
$\Rightarrow F = \bar{F}$

also $E \subset F$
$\Rightarrow E' \subset F'$
$\Rightarrow E \cup E' \subseteq F \cup F'$

$\Rightarrow \bar{E} \subseteq \bar{F} = F$

hence $\bar{E} \subseteq F$.

• **Theorem:** Let E be a non-empty subset of $\mathbb{R}$ which is bounded above. Let $y = \sup E$, then $y \in \bar{E}$.

Proof: Given that $y = \sup E$

as $r > 0$ and $y = \sup E$

$y - r$ is not $\sup E$

$\Rightarrow \exists$ some $x \in E$ st

$y - r < x \leq y$

$\Rightarrow x \in [y-r, y] \subseteq N_r(y)$

and $(y-r, y] \cap E \supseteq \{x\}$

$\Rightarrow N_r(y) \setminus \{y\} \cap E \neq \phi$

$\Rightarrow y$ is limit point of E.

$\Rightarrow y \in E' \subseteq \bar{E}$

• METRIC SUBSPACE - Let (X, d) be a metric space and $Y \subset X$ then (Y, d_Y) is metric subspace of (X, d) where d_Y is restriction map of metric d to $Y \times Y$.

24/10/21 Theorem: Suppose $Y \subseteq X$, then a subset $E \subset Y$ is open set in Y iff $E = Y \cap G$, for

$\Rightarrow \{q \in Y ; d(p,q) < r_p\} \subseteq E$

Let $V_p = \{q \in X ; d(p,q) < r_p\}$

define, $\bigcup_{p \in E} V_p = G$

then G is open in X [$\because$ Nbh are always open and arbitrary union of open sets is open]

Clearly, $E \subseteq G \cap Y$

$$G \cap Y = \left(\bigcup_{p \in E} V_p\right) \cap Y$$

$$= \bigcup_{p \in E} (V_p \cap Y) \subseteq E$$

Hence, $E = G \cap Y$.

Converse → If G is open set in X and $E = G \cap Y$.

To prove → E is open in Y.
Let $p \in E$
$\Rightarrow p \in G$.

and G is open in X, So $\exists\ r_p > 0$ such that $N_{r_p}(p) \subseteq E$

$$N_{r_p}(p) = \{q \in X ; d(p,q) < r_p\}$$

also $\{q \in Y ; d(p,q) < r_p\} \subseteq \{q \in X ; d(p,q) < r_p\} \subseteq E$

$\Rightarrow N_{r_p}(p) \cap Y \subseteq E$.

Corollary → Let $Y \subseteq X$ and $F \subseteq Y$. Then F is closed in Y iff $F = H \cap Y$ for some closed subset H of X.

Proof → Let F be closed in Y.

$\Rightarrow F^c$ is open in Y
$\Rightarrow F^c = G \cap Y$ (where G is open in X)
$\Rightarrow F = G^c \cap Y$ (Taking complement both sides).

Definition → Cover of Subset

Let (X,d) be a metric space. A family $\{A_\alpha ; \alpha \in \Lambda\}$ where Λ is non-empty set in X is called a cover of any subset A of X if $A \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha$.

Further if each A_α is open, then $\{A_\alpha ; \alpha \in \Lambda\}$ is called open cover of A.

If Λ is countable, then $\{A_\alpha ; \alpha \in \Lambda\}$ is called countable cover.

If Λ is finite, then $\{A_\alpha, \alpha \in \Lambda\}$ is called finite cover. → [means finite no. of sets].

Example :→ (1) $\mathbb{R} = \bigcup_{n=1}^{\infty} (-n,n)$ — open set

countable but union of countably many sets → (2) $\mathbb{R} = \bigcup_{n=1}^{\infty} [0,n] \cup [-n,0]$ → not open but closed set

uncountable → (3) $\mathbb{R} = \bigcup_{x \in \mathbb{R}} \left(x - \frac{1}{2}, x + \frac{1}{2}\right)$ → open set

Compact Metric Space :→

(1) A subset A of a metric space (x,d) is called compact if every open cover of A contains a finite subcover of A.

example→ $\mathbb{R} \subseteq \bigcup_{n \in \mathbb{N}} (-n,n)$ → Not compact set {finite no of sets doesnot formed}

of finite cover :→ $x \in \mathbb{R}$

$|[x]| + 1 = n$

$x \in (-n,n)$

for each family of open subsets $\{G_\alpha\}$ of X for which $A \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$, ∃ a finite subfamily say $\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}$ such that $A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$.

metric space X is called compact if X compact.

mple→ (1) $(0,1) = \bigcup_{n=2}^{\infty} \left(0, 1 - \frac{1}{n}\right)$

Doesnot have finite sub-cover

⇒ Not compact

(2) $(0,1) = \bigcup_{n=2}^{\infty} \left(\frac{1}{n}, 1\right)$

⇒ Not compact

Example of finite cover - Compact

$(0,1) = \bigcup_{n=2}^{\infty} \left(0, 1 - \frac{1}{n}\right) \cup \left(\frac{1}{n}, 1\right)$

$= \left[\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)\right] \cup \left[\left(0, \frac{2}{3}\right) \cup \left(\frac{1}{2}, 1\right)\right]$

Imp

25/10/21 **Theorem**→ Suppose $K \subset Y \subset X$. Then K is compact relative to Y iff K is compact relative to X.

Proof→ Let K be compact relative to X.

Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be an open cover of K in Y

⇒ $K \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$

as each V_α is open in Y

⇒ $V_\alpha = G_\alpha \cap Y$ for some open set G_α in X for $\forall \alpha$

⇒ $K \subseteq \bigcup_{\alpha \in \Lambda} (G_\alpha \cap Y)$

$\left(\bigcup_{\alpha \in \Lambda} G_\alpha\right) \cap Y \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$

$\Rightarrow \{G_\alpha\}_{\alpha \in \Lambda}$ is an open covering of K in X

as K is compact in X, $\exists\, G_{\alpha_1}, G_{\alpha_2}, \ldots G_{\alpha_n}$ st,

$K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots G_{\alpha_n}$

$\Rightarrow K \cap Y \subseteq (G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}) \cap Y$

$\Rightarrow K \subseteq \bigcup_{i=1}^{n} (G_{\alpha_i} \cap Y)$

$\Rightarrow K \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$

$\Rightarrow$ K is compact in Y.

Converse:- Now let K be compact in Y and let $\{G_\alpha\}_{\alpha \in \Lambda}$ be an open covering of K in X.

$\Rightarrow K \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$

$\Rightarrow K \cap Y \subseteq \left(\bigcup_{\alpha \in \Lambda} G_\alpha\right) \cap Y$

$\subseteq \bigcup_{\alpha \in \Lambda} (G_\alpha \cap Y)$

$= \bigcup_{\alpha \in \Lambda} V_\alpha$ (say)

then each V_α is open

$\Rightarrow \{V_\alpha\}_{\alpha \in \Lambda}$ is an open covering of K in Y.

$\Rightarrow \exists\, \alpha_1, \alpha_2, \ldots \alpha_n$ st $K \subseteq V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n}$

$\Rightarrow K \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$

$\Rightarrow K \subseteq \bigcup_{i=1}^{n} (G_{\alpha_i} \cap Y)$

$\Rightarrow K \subseteq \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap Y$

$\Rightarrow K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

$\Rightarrow$ K is compact in X.

Important

Theorem:- Compact subsets of metric spaces are closed.

Proof:- Let $K \subset X$ be compact.

To prove:- K is closed i.e., K^c is open.

Let $p \in K^c$ and $q \in K$

Let $0 < r < \frac{1}{2} d(p,q)$

Let $V_q = B(p,q)$ & $W_q = B(q,r)$

$x \in V_q \cap W_q$

$\Rightarrow d(x,p) < r$

& $d(x,q) < r$

$$d(p,q) \leq d(p,x) + d(x,q)$$
$$< r + r = 2r$$

$\rightarrow\leftarrow$

as $d(p,q) > 2r$

Now, $K \subseteq \bigcup_{q \in K} W_q$

$\Rightarrow \{W_q\}_{q \in K}$ is open covering of K.

and K is compact $\Rightarrow \exists q_1, q_2, q_3, \ldots q_n$ st

$$K \subseteq \bigcup_{i=1}^{n} W_{q_i} = W \text{ (say)}$$

Let $V_{q_1} \cap V_{q_2} \cap \ldots \cap V_{q_n} = V$

Now $V \cap W = \phi$ as $V_q \cap W_q = \phi \ \forall q \in K$

$\Rightarrow V \subseteq W^c = \cancel{\left(\bigcup_{i=1}^{n} W_{q_i}\right)^c = \bigcap} \subseteq K^c \ [\because K \subseteq W]$

$V \subseteq K^c$

$p \in V \Rightarrow p$ is interior point of K^c

$\Rightarrow K^c$ is open

Theorem → Closed subsets of compact sets are compact.

Proof → Let K be a compact set and $F \subset X$ such that F is closed.

To prove → F is compact

Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be an open cover of F

$$F \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$$

Now, $X = K \cup K^c$

$$K \subseteq X = F \cup F^c \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha \cup F^c$$
$$= \bigcup_{\alpha \in \Lambda} (V_\alpha \cup F^c)$$

$\Rightarrow \{V_\alpha \cup F^c\}_{\alpha \in \Lambda}$ is an open covering of K.

as K is compact, $\exists \alpha_1, \alpha_2, \ldots \alpha_n$ st

$$K \subseteq \bigcup_{i=1}^{n} (V_{\alpha_i} \cup F^c)$$
$$= \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \cup F^c$$

$\Rightarrow F \subseteq K \subseteq \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \cup F^c$

$\Rightarrow F \cap F \subseteq \left(\left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \cup F^c\right) \cap F$

$\Rightarrow F \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$

Corollary → If F is closed and K is compact then $F \cap K$ is compact.

Proof → $\underbrace{F \cap K}_{\text{closed}} \subseteq \underbrace{K}_{\text{compact}}$

Theorem → If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of K_α is non-empty, then $\bigcap_\alpha K_\alpha$ is non-empty.

Proof → Let $G_\alpha = K_\alpha^c$, then G_α is open $\forall \alpha$

Suppose $\bigcap_\alpha K_\alpha = \phi$

$$K_{\alpha_0} \cap \left(\bigcap_{\alpha \neq \alpha_0} K_\alpha\right) = \phi$$

$$\Rightarrow K_{\alpha_0} \subseteq \left(\bigcap_{\alpha \neq \alpha_0} K_\alpha\right)^c$$

$$\Rightarrow K_{\alpha_0} \subseteq \left(\bigcup_{\alpha \neq \alpha_0} K_\alpha^c\right)$$

$$\Rightarrow K_{\alpha_0} \subseteq \left(\bigcup_{\alpha \neq \alpha_0} G_\alpha\right)$$

$\Rightarrow \{G_\alpha\}_{\alpha \neq \alpha_0}$ is open covering of K_{α_0}

as K_{α_0} is compact, $\exists\ \alpha_1, \alpha_2, \dots \alpha_n$ st

$$K_{\alpha_0} \subseteq \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right)$$

$$= \left(\bigcup_{i=1}^{n} K_{\alpha_i}^c\right)$$

$$= \left(\bigcap_{i=1}^{n} K_{\alpha_i}\right)^c$$

$$\Rightarrow K_{\alpha_0} \cap \left(\bigcap_{i=1}^{n} K_{\alpha_i}\right) = \phi$$

which is contradiction.

Corollary → If $\{K_n\}$ is a sequence of non-empty compact sets st $K_n \supseteq K_{n+1}$ $\forall\ n = 1, 2, \dots$ then $\bigcap_{n=1}^{\infty} K_n \neq \phi$

Proof → Apply above theorem.

28/10/21 Theorem → Every infinite subset E of a compact set K has a limit point in K.

Proof → Assume that no point of K is a limit point of E

$\Rightarrow$ for each $q \in K$, $\exists$ a nbd $N(q)$ of q st.

$$N_q \setminus \{q\} \cap E = \phi$$

$$\Rightarrow N_q \cap E = \phi \text{ or } \{q\}$$

$\Rightarrow \boxed{K \subseteq \bigcup_{q \in K} N_q}$

$\Rightarrow \{N_q\}_{q \in K}$ is an open covering of K.

as K is compact, ∃ a finite subcovering

say $K \subseteq N_{q_1} \cup N_{q_2} \cup \cdots \cup N_{q_k}$.

$\Rightarrow K \cap E \subseteq (N_{q_1} \cup N_{q_2} \cup \cdots \cup N_{q_k}) \cap E$.

$\Rightarrow E \subseteq (N_{q_1} \cap E) \cup (N_{q_2} \cap E) \cup \cdots (N_{q_k} \cap E)$

$\subseteq \{q_1\} \cup \{q_2\} \cup \cdots \{q_k\}$

$= \{q_1, q_2, \cdots q_k\}$

$\Rightarrow$ E is finite

$\rightarrow\leftarrow$

$\Rightarrow$ E must have a limit point in K.

Theorem: If $\{I_n\}$ is a sequence of closed interval in R, st $I_n \supseteq I_{n+1}$, $n = 1, 2, \cdots$ then $\bigcap I_n \neq \phi$.

Proof: Let $I_n = [a_n, b_n]$ where $a_i, b_i \in \mathbb{R}$, $a_i < b_i$ $\forall i$.

then we get that

$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$

and a sequence $\{b_n\}$ st

$b_1 \geq b_2 \geq \cdots b_n \geq b_{n+1} \geq \cdots$

also sequence $\{a_n\}$ is bounded above by b_1 and $\{b_n\}$ is bounded below by a_1.

$\Rightarrow \{a_n\}$ converges to it LUB and $\{b_n\}$ converges to its ~~LUB~~ GLB.

Let L.U.B $\{a_n\} = x$

$\Rightarrow a_n \leq x \ \forall n$

also $a_m \leq b_m$

$\Rightarrow x = LUB \{a_n\} \leq b_m$

$\Rightarrow a_m \leq x \leq b_m$

$\Rightarrow x \in [a_m, b_m] \ \forall m$

$\Rightarrow x \in \bigcap I_n$

$\Rightarrow \bigcap I_n \neq 0$

Theorem: Let k be a +ve integer

If $I_n = [a_{n_1}, b_{n_1}] \times \cdots \times [a_{nk}, b_{nk}]$ $\longrightarrow$ K cell

and $I_n \supset I_{n+1}$ $\forall n = 1, 2, \cdots$

then $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

Proof: Let $I_{n,j} = [a_{nj} \times b_{nj}]$

then $I_{n,j} \supseteq I_{n+1,j}$.

By above theorem,

$$\bigcap_{n=1}^{\infty} I_{n,j} \neq \phi.$$

Let $x_j \in \bigcap_{n=1}^{\infty} I_{n,j}$

$\Rightarrow$ for each j, we set $x_j \in \bigcap_{n=1}^{\infty} I_{n,j} \; \forall \; 1 \le j \le k$

then taking $x^* = (x_1, x_2, \dots x_k)$

we see $x^* \in \bigcap_{n=1}^{\infty} I_{n,1} \times \bigcap_{n=1}^{\infty} I_{n,2} \times \dots \bigcap_{n=1}^{\infty} I_{n,k}$

$\Rightarrow x^* \in \bigcap_{n=1}^{\infty} (I_{n,1} \times I_{n,2} \times \dots I_{n,k})$

$\Rightarrow x^* \in \bigcap_{n=1}^{\infty} I_n$

<u>Theorem</u>- Every k-cell is compact.

<u>Proof</u>: Let I be a k-cell, consisting of the points.
$x = (x_1, x_2, \dots x_k)$ st $a_j \le x \le b_j \; \forall \; 1 \le j \le k$

Put $\delta = \sqrt{\sum_{j=1}^{k} (b_j - a_j)^2}$ [illegible]

i.e., If $x, y \in I$, $\|x - y\| < \delta$.

Suppose I is not compact, then $\exists$ an open cover $\{G_\alpha\}$ of I, which does not have a finite subcover.

Put $c_j = \frac{a_j + b_j}{2}$

Then, the intervals $[a_j, c_j]$ and $[c_j, b_j]$ determine 2^k, k-cells, Q_j whose union is I.

Now, atleast one of these small k-cell is not covered by finitely many G_α's

(Call it I_1)
Now we subdivide I_1 in the same way as we did for I and continue this process

We get a sequence $\{I_n\}$ st

(i) $I \supseteq I_1 \supseteq I_2 \supseteq \dots$
(ii) I_n is not covered by finitely many G_α's.
(iii) If $x, y \in I_n$ then $\|x - y\| < \frac{\delta}{2^n}$

by previous theorem $\bigcap_{n=1}^{\infty} I_n \neq \phi$

say $x^* \in \bigcap_{n=1}^{\infty} I_n$

$x^* \in I \subseteq \bigcup G_\alpha$

$\Rightarrow \exists$ some α_0 st $x^* \in G_{\alpha_0}$
as G_{α_0} is open, $\exists \; r > 0$ st

$N_r(x^*) \subseteq G_{\alpha_0}$

Choose, n to be large enough st

$\frac{\delta}{2^n} < r$

$N_r(x^*) = \{y : \|x^* - y\| < r\}$

$\Rightarrow$ ~~$x, y \in I_n$ st $\|x - y\| \le \frac{\delta}{2^n}$~~

If $x \in I_n$, then $\|x^* - x\| < \frac{\delta}{2^n} < r$

$\Rightarrow x \in N_r(x^*)$

$\Rightarrow I_n \subseteq N_r(x^*) \subseteq G_{\alpha_0}$

$\rightarrow\leftarrow$

Our assumption is wrong

$\therefore$ K-cell is compact.

10/2 **Theorem:** If a set $E \subseteq R^k$ has one of the three properties, then it has the other two.

(1) E is closed and bounded
(2) E is compact
(3) Every infinite subset of E has a lt. point in E.

Proof: (1) $\Rightarrow$ (2)
as E is bounded, $\exists$ a K-cell I st $E \subseteq I$
as I is compact and E is closed.

(3) $\Rightarrow$ (1) Suppose E is not closed.

$\Rightarrow \exists$ some $x_0 \in R^k$ st x_0 is limit point of E and $x_0 \notin E$

$\Rightarrow B\left(x_0, \frac{1}{n}\right) \cap E \neq \phi$ for each $n \in N$.

If $x_n \in B\left(x_0, \frac{1}{n}\right) \cap E$, then we get an infinite set
$S = \{x_1, x_2, \ldots, x_n, \ldots\}$

By (3), S has a limit point in E.

$B(x_0, r) / \{x_0\} \cap S \neq \phi$

$\Rightarrow x_0$ is a limit point of S in E.

Let $y (\neq x_0)$ be limit point of S.

$\Rightarrow B(y, r) \setminus \{y\} \cap S \neq \phi \ \forall r > 0$.

~~$d(x_n, y) \le d(x_n, x_0) + d(x_0, y)$~~

$d(x_n, y) + d(x_n, x_0) \geq d(x_0, y)$

$\Rightarrow d(x_n, y) \geq d(x_0, y) - d(x_n, x_0)$

$\geq d(x_0, y) - \frac{1}{n}$

$\geq \frac{1}{2} d(x_0, y) = c$ (say).

for sufficient large N

for $n \geq N$
$d(x_n, y) \geq c$

Our cell is closed.

Assume that E is not bounded

$\Rightarrow$ for each $n \in \mathbb{N}$ $\exists$ $x_n \in E$ st,

$\|x_n\| > n$

then the set $S\{x_1, x_2, \ldots x_n, \ldots\}$ is infinite subset of E.

$\Rightarrow$ By given condition, S has a limit pt in E (say x^*)

$\Rightarrow B(x^*, \frac{1}{2}) \cap S$ = infinite subset of S.

$\Rightarrow d(x_n, x^*) < \frac{1}{2}$ for infinitely many elements of S.

$\Rightarrow \|x_n - x^*\| < \frac{1}{2}$ for infinitely many elements of S

$\Rightarrow \|x_n\| = \|x_n - x^* + x^* - 0\| \leq \|x_n - x^*\| + \|x^*\|$

$n < \|x_n\| \leq \frac{1}{2} + \|x^*\|$

$\Rightarrow \|x^*\| \geq n - \frac{1}{2}$ for infinitely many n.

$\Rightarrow \|x^*\| \geq n-1$ for infinitely many n

$x^* \notin S$ which is contradiction

$\therefore E$ is a bounded set

Imp

Heine Borel Theorem $\rightarrow$ A subset $E \subseteq R^k$ is compact iff it is closed and bounded. $(1 \Rightarrow 2, 3 \Rightarrow 1)$, $(2 \Rightarrow 3, 3 \Rightarrow 1)$

Imp

Bolzano Weierstrass Theorem $\rightarrow$ Every bounded infinite subset E of R^k has a limit point in E.

Proof $\rightarrow$ Let E be bounded infinite subset of R^k

$\Rightarrow E \subseteq I$, for some k-cell I.

as I is compact and E is infinite subset of I, E must have a limit point in E. (Equivalent conditions) $(2 \Rightarrow 3)$

Separated Sets $\rightarrow$ Two sets A and B are called separated if

$A \cap \bar{B} = \phi$ and $B \cap \bar{A} = \phi$

EXAMPLES $\rightarrow$

~~$A = (0,1)$ $\bar{A} = [0,1]$~~

$A = (0,2)$
$B = (2,3)$

$\bar{A} = [0,2]$
$\bar{B} = [2,3]$

· Two disjoint ... disjoint
· Separated may be ...

(3) $A = (0,2]$, $B = (2,3)$
$\bar{A} = [0,2]$, $\bar{B} = [2,3]$

Not separated $(\because B \cap \bar{A} \neq \phi)$

02/11/21 **Connected Sets:-** A set $E \subset X$ is called connected if it can not be written as union of two non-empty separated sets.

Otherwise the set is called disconnected sets. $\boxed{E = A \cup B,\ A \neq \phi,\ B \neq \phi}$ (separate kar sakte)

Theorem: A subset E of $\mathbb{R}$ is connected iff it has the following property
If $x, y \in E$ and $x < z < y$ then $z \in E$.
i.e. the only connected sets in $\mathbb{R}$ are the intervals (open or closed)

Let $E \subseteq \mathbb{R}$ be a connected set and $x, y \in E$ st $\exists z$, $x < z < y$ but $z \notin E$.

Let $\boxed{\begin{array}{l} A_z = E \cap (-\infty, z) \\ B_z = E \cap (z, \infty) \end{array}}$

then, $E = A_z \cup B_z$

as $\boxed{x \in A_z \text{ and } y \in B_z}$

$A_z \neq \phi$, $B_z \neq \phi$

$\bar{A_z} \cap B_z = \overline{(E \cap (-\infty, z))} \cap B_z$
$\subseteq \overline{(-\infty, z)} \cap B_z$
$= (-\infty, z] \cap B_z$
$\subseteq (-\infty, z] \cap (z, \infty)$
$= \phi$

Similarly, $A_z \cap \bar{B_z} = \phi$

$\Rightarrow E$ is disconnected.

$\rightarrow\leftarrow$ (which is contradiction)

Conversely: Suppose whenever $x, y \in E$ st $x < z < y$ then $z \in E$

and E is disconnected.

$\Rightarrow E = A \cup B$, $A \neq \phi$, $B \neq \phi$

$\bar{A} \cap B = \phi$, $A \cap \bar{B} = \phi$

Let $x \in A$, $y \in B$

Let $\boxed{z = \sup \{A \cap [x, y]\}}$

As $[x, y]$ is bounded

$\Rightarrow A \cap [x, y]$ is bounded, so supremum exists.

$\Rightarrow z \in \bar{A}$

but $\bar{A} \cap B = \phi$

$\Rightarrow z \notin B$

Case 1 → If $z \notin A$

$\Rightarrow z \notin A \cup B = E$

then $x < z < y$

such that $z \notin E$ but $x, y \in E$.

→← (which is contradiction)

Case 2 → If $z \in A$

$\Rightarrow z \notin \bar{B}$

Imp point

$\Rightarrow \exists$ some z_1 st $z < z_1 < y$ and $z_1 \notin B$

$x < z < z_1 < y$

$z_1 \notin A$ ($\because z$ is the sup $(A \cap [x,y])$)

$\Rightarrow z_1 \notin E$

which is contradiction.

SEQUENCE : Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is a map $f: \mathbb{N} \to X$ defined by -

$f(n) = x_n$

Convergence: We say a sequence $\{x_n\}$ in X converges to a point $x \in X$ if for given $\epsilon > 0$, $\exists N \in \mathbb{N}$ st

$d(x_n, x) < \epsilon \;\; \forall n \geq N$

and we write as $\lim_{n \to \infty} x_n = x$

Imp

Theorem: Let $\{x_n\}$ be a sequence in a metric space, then

(1) $\{x_n\}$ converges to $x \in X$, iff every nbd of x contains all but finitely many terms of $\{x_n\}$

(2) If $x, x' \in X$ and $x_n \to x$, $x_n \to x'$ then $x = x'$

(3) If $\{x_n\}$ is convergent, then $\{x_n\}$ is bdd.

(4) If $E \subseteq X$ and $x \in E'$ then there is a sequence $\{x_n\}$ in E st $x = \lim_{n \to \infty} x_n$

Proof: (1) Let $x_n \to x$ and V be some nbd. of x

$V = B(x, r)$; $r > 0$

$\Rightarrow \exists$ some $N \in \mathbb{N}$ st $d(x_n, x) < r \;\; \forall n \geq N$

$\Rightarrow x_N, x_{N+1}, \ldots \in B(x, r)$

$\Rightarrow$ all but finitely many terms of $\{x_n\}$ belong to V.

Converse - Let $\epsilon > 0$ be given

Let $V = B(x, \epsilon)$

Let V contains all terms of $\{x_n\}$ except say $x_1, x_2, \ldots x_N$

$\Rightarrow x_n \in V \;\; \forall n \geq N$

$\Rightarrow d(x_n, x) < \epsilon \;\; \forall n \geq N \Rightarrow \{x_n\} \to x$.

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Proof: (2) As $x_n \to x \Rightarrow$ for given $\epsilon > 0$ $\exists N_1 \in \mathbb{N}$ st $d(x_n, x) < \epsilon$ $\forall$ $n \geq N_1$

As $x_n \to x' \Rightarrow$ for $\epsilon > 0$, $\exists N_2 \in \mathbb{N}$ st. $d(x_n, x') < \epsilon$ $\forall$ $n \geq N_2$

Let $N = \max\{N_1, N_2\}$

~~then $\forall$ n, m $\geq$ N~~

$\cancel{d(x_n, x_m) \leq d(x_n, x) + d(x_m, x)}$

$\cancel{\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon}$

$d(x, x') \leq d(x, x_n) + d(x_n, x')$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$ $\forall$ $n \geq N$

$= \epsilon$.

but ϵ was arbitrary.

$\Rightarrow d(x, x') = 0$

$\Rightarrow \boxed{x = x'}$

$\{x_n\}$ converges, then $\{x_n\}$ is bounded.

Given $\epsilon > 0$ $\exists N \in \mathbb{N}$ st

$d(x_n, x) < \epsilon$ $\forall$ $n \geq N$

$\max\{d(x_1, x), d(x_2, x), \ldots d(x_{N-1}, x), \epsilon\}$

$\Rightarrow d(x_n, x) \leq r$ $\forall$ n

$\exists x_n \in B(x, r)$ $\forall$ n

$\Rightarrow \{x_n\}$ is bounded.

(4) If $E \subseteq X$ & $x \in E'$, then $\exists$ a sequence $\{x_n\}$ in E such that $x = \lim_{n \to \infty} x_n$.

Proof: As $x \in E'$, $B(x, \frac{1}{n}) \cap E \neq \phi$ $\forall n \in \mathbb{N}$

Let $x_n \in B(x, \frac{1}{n}) \cap E$ and $x_n \neq x$.

then $\{x_n\}$ is a sequence in E

Let $\epsilon > 0$ then by Archimedian property

$\exists$ a natural number N such that

$N > \frac{1}{\epsilon}$ $\Rightarrow$ $\epsilon > \frac{1}{N}$

$\forall$ $n \geq N$

$\Rightarrow \frac{1}{n} \leq \frac{1}{N} < \epsilon$.

$d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \epsilon$.

$\Rightarrow x_n \to x$

Subsequence :- Let $\{x_n\}$ be a sequence. If $n_1 < n_2 < n_3 < \dots$ be a sequence of naturals, then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

1/11/21 Note :- If $x_n \to x$, then every subsequence of $\{x_n\}$ also converges to x. but not conversely

example :- $1, -1, 1, -1, 1, -1 \dots$

Subsequence $1, 1, 1 \dots$
$\Rightarrow 1$

Subsequential limits :- The limits of subsequences of a sequence are called subsequential limits.

Theorem :- (a) If $\{x_n\}$ is a sequence in a compact metric space, then some subsequence of $\{x_n\}$ converges to a point of X.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Proof :- (a) Let E be the range set of sequence $\{x_n\}$

Case 1 - If E is finite

$\Rightarrow \exists$ some $x \in E$ such that x appears infinitely many times in $\{x_n\}$

take the subsequence $\{x_{n_k}\} = \{x\}$

then $x_{n_k} \to x \in E \subseteq X$.

Case 2 - If E is infinite

and $E \subseteq X$, X is compact.

$\Rightarrow E$ must have a limit point in E say 'x'.

Choose n_1 such that $d(x, x_{n_1}) < \frac{1}{1}$.

Choose n_2 such that $d(x, x_{n_2}) < \frac{1}{2}$.

where $n_2 > n_1$

We see that, we can choose n_i's such that $d(x, x_{n_i}) < \frac{1}{i}$

and $n_i > n_{i-1} > \dots > n_2 > n_1$

Let $\epsilon > 0$ be given we can choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$

and then $d(x_{n_k}, x) < \frac{1}{k} \leq \frac{1}{N} < \epsilon. \quad \forall k \geq N$

$\Rightarrow \{x_{n_k}\} \longrightarrow x$

(b) Every bounded set of $\mathbb{R}^k$ lies in a k-cell and every k-cell is compact.

$\Rightarrow$ If sequence $\{x_n\}$ is bounded then $\{x_n\}$ is contained in a k-cell which is compact.

So by (a) sequence must converge to some point of X.

Cauchy sequence $\rightarrow$ A sequence $\{x_n\}$ in a metric space X is called cauchy if for given $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ st $d(x_n, x_m) < \epsilon$ $\forall$ $n, m \geq N$.

Diameter of a set $\rightarrow$ Let X be a metric space and $E \subseteq X$.

$S = \{d(x, y) ; x, y \in E\}$

$$\boxed{\operatorname{diam} E = \sup S}$$

Remark $\rightarrow$ If $\{x_n\}$ is cauchy sequence in X and $E_N = \{x_N, x_{N+1}, \ldots\}$ then $\lim_{n \to \infty} \operatorname{diam} E_N = 0$. $\underbrace{}_{\text{tail of sequence}}$

Conversely, if $\lim_{N \to \infty} \operatorname{diam} E_N = 0$

$\{x_n\}$ is a Cauchy seq. in X.

Theorem $\rightarrow$ (a) Let $E \subseteq X$, then $\operatorname{diam} E = \operatorname{diam} \bar{E}$

(b) If $\{K_n\}$ is a sequence of compact sets in X such that $K_n \supseteq K_{n+1}$, for $n = 1, 2, 3$ and if $\lim_{n \to \infty} \operatorname{diam} K_n = 0$ then $\bigcap_{n=1}^{\infty} K_n$ consists of only one point

Proof $\rightarrow$ As $E \subseteq \bar{E}$

$\operatorname{diam} E = \sup \{d(x, y) : x, y \in E\}$

$\operatorname{diam} \bar{E} = \sup \{d(x, y) : x, y \in \bar{E}\}$

$\Rightarrow \operatorname{diam} E \leq \operatorname{diam} \bar{E}$

If $x, y \in \bar{E}$ and $\epsilon > 0$ be given then $\exists$ $x', y' \in E$ such that

$d(x, x') < \frac{\epsilon}{2}$ and $d(y, y') < \frac{\epsilon}{2}$

$\Rightarrow d(x, y) \leq d(x, x') + d(x', y)$
$\leq d(x, x') + d(x', y') + d(y', y)$

$< \frac{\epsilon}{2} + d(x', y') + \frac{\epsilon}{2}$

$= d(x', y') + \epsilon$

$\Rightarrow d(x, y) \leq d(x', y') + \epsilon \leq \operatorname{diam}(E) + \epsilon$

taking sup over all $x, y \in \bar{E}$

$\operatorname{diam} \bar{E} \leq \operatorname{diam} E + \epsilon$

as $\epsilon > 0$ is arbitrary, ... $\leq$ diam E

(b) Let $K = \bigcap_{n=1}^{\infty} K_n$

and assume that K has more than one point

$\Rightarrow$ diam $K > 0$

and $K \subseteq K_n \ \forall n$

$\Rightarrow$ diam $(K) \leq$ diam $(K_n) \ \forall n$

$\Rightarrow \lim_{n \to \infty} K_n > 0$

$\rightarrow\leftarrow$

which is contradiction.

Hence K has only 1 point.

Theorem: (a) In any metric space, every convergent sequence is cauchy.

(b) If X is compact metric space and $\{x_n\}$ is a cauchy sequence in X, then $\{x_n\}$ converges to some point of X.

(c) In $\mathbb{R}^k$, every cauchy sequence converges

Proof: (a) Let $x_n \to x$ and $\epsilon > 0$ be given

$\Rightarrow \exists\, N \in \mathbb{N}$ such that
$d(x_n, x) < \frac{\epsilon}{2} \quad \forall\, n \geq N.$

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall\, m, n \geq N$$
$$= \epsilon \quad \forall\, n, m \geq N$$

$\{x_n\}$ is cauchy sequence

But converse is not true (Rationals)

(b) Let X be compact metric space and $\{x_n\}$ be cauchy.

Let $E_N = \{x_N, x_{N+1}, \ldots\}$

$\Rightarrow$ diam $E_N = 0$ as $N \to \infty$

$E_N \supseteq E_{N+1} \Rightarrow \overline{E_N} \supseteq \overline{E_{N+1}}$

$\Rightarrow \bigcap_{N=1}^{\infty} \overline{E_N}$ has only one element (by previous thm (b))

Let $\{x\} = \bigcap_{N=1}^{\infty} \overline{E_N}$

Claim: $x_n \to x$.

Let $\epsilon > 0$ be given
as diam $\overline{E_N} = 0$ as $N \to \infty$

$\Rightarrow \exists\, N_0$ st $\forall\, N \geq N_0$

$|$ diam $\overline{E_N} - 0 | < \epsilon$

$\Rightarrow \operatorname{diam} \overline{E_N} < \epsilon$

$\Rightarrow d(x_n, x) < \epsilon \;\; \forall \; n \geq N_0$

$\Rightarrow x_n \to x$

(c) Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^k$ then $\lim\limits_{N\to\infty} \operatorname{diam} E_N = 0$.

where $E_N = \{x_N, x_{N+1}, \dots\}$

given $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that

$\operatorname{diam} E_n < \epsilon \;\; \forall \; n \geq N$

take $\epsilon = 1$

$\Rightarrow$ $\exists$ N_1 such that $\operatorname{diam} E_N \leq 1 \;\; \forall \; N \geq N_1$

range of $\{x_n\} \subseteq E_{N_1} \cup \{x_1, x_2, \dots, x_{N-1}\}$

$\Rightarrow \{x_n\}$ is bounded

$\Rightarrow \{x_n\}$ can be contained in a K-cell.

as every K-cell is compact

$\{x_n\}$ is convergent by (b)

Unit 02

Limit and Continuity

Definition $\to$ Let X and Y be two metric spaces. Suppose $E \subseteq X$, then if c is a limit point of E. Let $f: E \to Y$

We write, $\lim\limits_{x\to c} f(x) = l$ if for given $\epsilon > 0$, $\exists$ $\delta > 0$ such that $d_Y(f(x), l) < \epsilon$ whenever $x \in E$, $0 < d_X(x, c) < \delta$ i.e.,

$f(x) \in B_Y(l, \epsilon)$ whenever $x \in B_X(c, \delta) \cap E$, $x \neq c$.

NOTE $\to$ c may or may not belong to E but $c \in E'$.

Theorem $\to$ Let X, Y, E, f, c be as in defn. then $\lim\limits_{x\to c} f(x) = l$ iff $\lim\limits_{x\to\infty} f(x_n) = l$ for every sequence $\{x_n\}$ in E st $x_n \neq c$ and $\lim\limits_{n\to\infty} x_n = c$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \cos\frac{1}{x}$, Let $x_n = \frac{1}{n\pi} \to 0$, $f(x_n) = \cos n\pi$

Proof: Suppose $\lim\limits_{x\to c} f(x) = l$ —— (1)

and let $\{x_n\}$ be a sequence in E such that $x_n \neq c$ and $\lim\limits_{n\to\infty} x_n = c$ —— (2)

(1) $\Rightarrow$ given $\epsilon > 0$, $\exists$ $\delta > 0$ such that

Limit and Continuity

Definition → Let X and Y be two metric spaces. Suppose $E \subseteq X$, then if c is a limit point of E. Let $f : E \to Y$

We write, $\lim_{x \to c} f(x) = l$ if for given $\epsilon > 0$, $\exists\ \delta > 0$ such that $d_Y(f(x), l) < \epsilon$ whenever $x \in E$, $0 < d_X(x, c) < \delta$ i.e.,

$f(x) \in B_Y(l, \epsilon)$ whenever $x \in B_X(c, \delta) \cap E$, $x \neq c$.

NOTE → c may or may not belong to E but $c \in E'$.

Theorem → Let X, Y, E, f, c be as in def^n,

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = \cos\frac{1}{x}$

Let $x_n = \frac{1}{n\pi} \to 0$

$f(x_n) = \cos n\pi$

then $\lim_{x \to c} f(x) = l$ iff $\lim_{x \to \infty} f(x_n) = l$ for every sequence $\{x_n\}$ in E st $x_n \neq c$ and $\lim_{n \to \infty} x_n = c$

Proof: Suppose $\lim_{x \to c} f(x) = l$ —— ①

and let $\{x_n\}$ be a sequence in E such that $x_n \neq c$ and $\lim_{n \to \infty} x_n = c$ —— ②

① ⇒ given $\epsilon > 0$, $\exists\ \delta > 0$ such that

$d_Y(f(x), l) < \epsilon$ whenever $0 < d_X(x, c) < \delta$ — ③

Also ② ⇒ for $\delta > 0$ $\exists N \in \mathbb{N}$ such that

$0 < d_X(x_n, c) < \delta \quad \forall n \geq N$ — ④

From ③ and ④, $d_Y(f(x_n), l) < \epsilon \quad \forall n \geq N$

$\Rightarrow \lim_{n\to\infty} f(x_n) = l$

Converse → To prove – $\lim_{x\to c} f(x) = l$

Assume that $\lim_{x\to c} f(x) \neq l$

⇒ ∃ atleast one $\epsilon > 0$ such that for every $\delta > 0$.

$d_Y(f(x), l) \geq \epsilon$ whenever $0 < d_X(x, c) < \delta$.

take $\delta_n = \frac{1}{n}$

⇒ $\exists x_n \in E$ such that $0 < d_X(x_n, c) < \frac{1}{n}$

but $d_Y(f(x_n), l) \geq \epsilon$

⇒ $\lim_{n\to\infty} x_n = c$ but $\lim_{n\to\infty} f(x_n) \neq l$

→← (which is contradiction)

So, our assumption is wrong

$\lim_{x\to c} f(x_n) = l$.

Continuity of function → Suppose X and Y are metric spaces. $E \subseteq X$, $c \in E$

$f: E \to Y$ then f is said to be continuous at c if given $\epsilon > 0$, $\exists \delta > 0$ such that

$d_Y(f(x), f(c)) < \epsilon$ whenever $d_X(x, c) < \delta$ $\forall x \in E$.

NOTE → If c is isolated point of E, then f is continuous at c.

Proof: As c is isolated point ∃ a nbd of c such that $N_\delta(c) \cap E = ~~\phi~~ \{c\}$

$\Rightarrow d_Y(f(x), f(c)) = d_Y(f(c), f(c)) = 0 < \epsilon$

for every ϵ.

⇒ f is continuous at c.

Theorem → ~~If~~ Let X, Y, Z be metric spaces $E \subseteq X$. $f: E \to Y$, $g: f(E) \to Z$, $h: E \to Z$ defined by $h(x) = g(f(x))$ $\forall x \in E$. If f is continuous at $c \in E$, g is continuous at $f(c) \in Y$ then h is continuous at $x = c$.

Proof: Let $\epsilon_1 > 0$ be given, since g is continuous at $f(c)$.

~~$\exists N \in \mathbb{N}$ such that~~ $\exists \delta_1 > 0$ st

$d_Z(g(f(c)), g(y)) < \epsilon_1$ whenever $d_Y(f(c), y) < \delta_1$ — ①

As f is continuous at c, $\exists\, \delta_2 > 0$ such that

$d_Y(f(c), f(x)) < \epsilon$ whenever $d_X(c, x) < \delta_2$ —— (2)

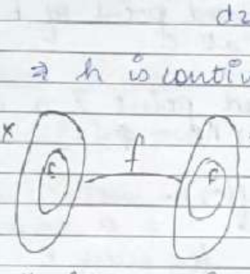
From (1) and (2), we get,

$d_X(c, x) < \delta_2 \Rightarrow d_Z(g(f(c)), g(f(x))) < \epsilon_1$

$d_Z(h(c), h(x)) < \epsilon_1$

$\Rightarrow h$ is continuous at c.

27/11/21 (1)



$f: X \to Y$ is a fxn

$F \subseteq X$, $f(E) = F$

$f^{-1}(F) = \{x \in X \,;\, f(x) \in F\}$

$X = \{1,2,3,4,5\}$ $\boxed{f^{-1}(f(E)) \supseteq E}$ (1-1)

$Y = \{a, b\}$

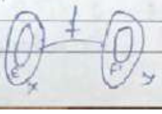
$f(x) = a$, $E = \{1, 2\}$ equality holds if function is one-one

$f(1) = \{a\} = F$

$f^{-1}(F) = X \supseteq E$

(2) $f(f^{-1}(F')) \subseteq F'$ (Onto)

$f: X \to Y$ is a fxn



$E \subseteq X$, $f(E) = F$, $F' \subseteq Y$

$X = \{1,2,3\}$ $Y = \{a, b, c, d\}$

$F' = \{c, d\}$ equality holds if fxn is onto

$f^{-1}(F') = \{3\}$

$f(f^{-1}(F')) = \{c\}$

$f: X \to Y$ is a fxn

Theorem: (1) $f^{-1}(f(E)) \supseteq E$

(2) $f(f^{-1}(F')) \subseteq F'$

Proof: (1) Let $x \in E$

$\Rightarrow f(x) \in f(E)$

$\Rightarrow x \in f^{-1}(f(E))$

$\Rightarrow E \subseteq f^{-1}(f(E))$ —— (1)

Now let f be 1-1

Let $x \in f^{-1}(f(E))$

$\Rightarrow f(x) \in f(E)$

$\Rightarrow f(x) = f(y)$ for some $y \in E$

$\Rightarrow x = y \in E$

$\Rightarrow x \in E$

$f^{-1}(f(E)) \subseteq E$ —— (2) From 1 and 2

$E = f^{-1}(f(E))$

② $f(f^{-1}(F')) \subseteq F'$

Let $y \in f(f^{-1}(F'))$

$\Rightarrow y = f(x)$, for some $x \in f^{-1}(F')$

$\Rightarrow f(x) \in Y$, $f(x) \in F'$

$\Rightarrow y \in F'$

$\Rightarrow f(f^{-1}(F')) \subseteq F'$

Now let f be onto

Let $y \in F'$

$\Rightarrow \exists\ x \in X$ st $y = f(x)$

$\Rightarrow x = f^{-1}(y) \in f^{-1}(F')$

$\Rightarrow f(x) \in f(f^{-1}(F'))$

$\Rightarrow y \in f(f^{-1}(F'))$

Imp.

Theorem: A mapping f of a metric space X into a metric space Y is continuous on X iff $f^{-1}(V)$ is open in X for every open set V in Y.

Proof: Firstly assume that $f: X \to Y$ is continuous and let V be open set in Y.

Let $x_0 \in f^{-1}(V)$

$\Rightarrow f(x_0) \in V$

As V is open in Y, $\exists\ \epsilon > 0$ st

$$B(f(x_0), \epsilon) \subseteq V \quad \text{--- } ①$$

Since f is continuous, for $\epsilon > 0\ \exists\ \delta > 0$ such that.

$$d_Y(f(x), f(x_0)) < \epsilon \text{ whenever } \underbrace{d_X(x, x_0) < \delta}_{②}$$

① $\Rightarrow d_Y(f(x_0), v) < \epsilon \ \forall\ v \in V$

② $\Rightarrow$ If $x \in B(x_0, \delta)$, then $f(x) \in B(f(x_0), \epsilon)$

$\Rightarrow f(B(x_0, \delta)) \subseteq B(f(x_0), \epsilon)$

$\Rightarrow B(x_0, \delta) \subseteq f^{-1}(B(f(x_0)), \epsilon)$

$\Rightarrow B(x_0, \delta) \subseteq f^{-1}(V)$

$\Rightarrow x_0$ is interior point of $f^{-1}(V)$

$\Rightarrow f^{-1}(V)$ is open.

Conversely: Let $x_0 \in X$

TP: f is continuous at x_0.

Let $\epsilon > 0$ be given

Let $V = B_Y(f(x_0), \epsilon)$

$\Rightarrow V$ is open in Y

also, $f^{-1}(V)$ is open in X as $f(x_0) \in V$
$\exists\ \delta > 0$ st $\qquad \Rightarrow x_0 \in f^{-1}(V)$

$$B_X(x_0, \delta) \subseteq f^{-1}(V)$$
$$= f^{-1}\left(B_Y(f(x_0), \epsilon)\right)$$

$\Rightarrow$ If $x \in B_X(x_0, \delta)$

$\Rightarrow x \in f^{-1}(V)$
$\Rightarrow f(x) \in V$
$= B_Y(f(x_0), \epsilon)$

$\Rightarrow d_Y(f(x), f(x_0)) < \epsilon$

$\Rightarrow f$ is continuous at x_0

9/11/21 # $f: R \to R$

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f^{-1}(0,1) = \{x \in R \ ; f(x) \in (0,1)\}$$
$$= \{x \in R \ ; \frac{1}{x} \in (0,1)\}$$

Example: $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$
$E = (-2, 2)$
$f(E) = [0, 4)$

$0 < \frac{1}{x} < 1 \qquad \Rightarrow 0 < 1 < x$

$\Rightarrow x > 0 \qquad \Rightarrow x > 1$
$(1, \infty)$

$$f^{-1}(0,1) = (1, \infty)$$

$f: R \to R$

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 1/2 & \text{if } x = 0 \end{cases}$$

$$f^{-1}(0,1) = \{x \in \mathbb{R} \ ; \frac{1}{x} \in (0,1)\} \cup \{0\}$$

$x > 1 \qquad (1, \infty) \cup \{0\}$

$f^{-1}(0,1) = (1, \infty) \cup \{0\}$ (This is not continuous)

Corollary → $f: X \to Y$ is continuous iff $f^{-1}(F)$ is closed in X for every closed set F in Y.

Proof: $f^{-1}(F)$ is closed $\Leftrightarrow (f^{-1}(F))^c$ is open
$\Leftrightarrow f^{-1}(F^c)$ is open
$\Leftrightarrow F^c$ is open [∵ of previous theorem]
$\Leftrightarrow F$ is closed

Theorem: Let X and Y be metric spaces and $f: X \to Y$ then the following conditions are equivalent -

(1) f is continuous on X

(2) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \quad \forall\, B \subseteq Y$

(3) $f(\overline{A}) \subseteq \overline{f(A)} \quad \forall\, A \subseteq X$

Proof: $1 \Rightarrow 2$ Let f be continuous on X and $B \subseteq Y$.

$\Rightarrow \overline{B}$ is closed in Y

$\Rightarrow f^{-1}(\overline{B})$ is closed in X

Now, $B \subseteq \overline{B}$

$f^{-1}(B) \subseteq f^{-1}(\overline{B})$

$\overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$

$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$

$2 \Rightarrow 3$ Let $A \subseteq X$ and $B = f(A)$

then $A \subseteq f^{-1}(f(A))$

$= f^{-1}(B)$

$\overline{A} \subseteq \overline{f^{-1}(B)}$

$\overline{A} \subseteq f^{-1}(\overline{B})$ (by (2))

$f(\overline{A}) \subseteq f(f^{-1}(\overline{B}))$
$\subseteq \overline{B} = \overline{f(A)}$

$3 \Rightarrow 1$ Let $F \subseteq Y$ such that F is closed

To prove ~~E~~ $E = f^{-1}(F)$ is closed in X

$\Rightarrow f(\overline{E}) \subseteq \overline{f(E)}$ — By (3)

$= \overline{f(f^{-1}(F))} \subseteq \overline{F} = F$

$\Rightarrow f(\overline{E}) \subseteq F$

$\overline{E} \subseteq f^{-1}(f(\overline{E})) \subseteq f^{-1}(F) = E$

also $E \subseteq \overline{E}$

$\Rightarrow E = \overline{E}$

$\Rightarrow E$ is closed.

Theorem: Suppose f is continuous fn ~~of~~ a compact metric space X to a metric space Y. Then $f(X)$ is also compact.

Proof: $f: X \to Y$
$f(X) \subseteq Y$

Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be an open covering of $f(X)$

$\Rightarrow f(X) \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$

$\boxed{X = f^{-1}(f(X))}$ $\quad [f^{-1}(f(X)) \supseteq X]$

$\subseteq f^{-1}\left(\bigcup_{\alpha\in\Lambda} V_\alpha\right)$

$= \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$

$\Rightarrow X \subseteq \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$

$\{f^{-1}(V_\alpha)\}_{\alpha\in\Lambda}$ is open ~~open~~ covering in X.

[$\because$ f is continuous and V_α's are open $\Rightarrow f^{-1}(V_\alpha)$'s are also open in X]

as X is compact
$\exists$ $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ st

$X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$

$\Rightarrow f(X) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right)$

$\Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(V_{\alpha_i}))$

$\subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$

$\therefore f(X)$ is compact.

Bounded functions: A mapping $f: E \to \mathbb{R}^k$ is said to be bounded if $\exists$ $M \in \mathbb{R}$ st $\|f(x)\| \leq M$ $\forall$ $x \in E$.

Theorem: Let f be a continuous function of a compact metric space X into $\mathbb{R}^k$, then $f(X)$ is always bounded thus f is bounded.

Proof: As f is continuous and X is compact

$\Rightarrow f(X) \subseteq \mathbb{R}^k$ is also compact

$\Rightarrow f(X)$ is bounded in $\mathbb{R}^k$

$\Rightarrow f$ is bounded.

sup

Theorem: Suppose f is continuous real function on a compact metric space X and $M = \sup_{x\in X} f(x)$, $m = \inf_{x\in X} f(x)$ then $\exists$ points x_0, x_1 in X st $M = f(x_0)$ and $m = f(x_1)$ i.e. $\boxed{M = \max_{x\in X} f(x)}$, $\boxed{m = \min_{x\in X} f(x)}$.

Proof: $M = \sup_{x\in X} f(x)$ is a limit point of $f(x)$ then $M \in \overline{f(X)}$.

Also, $f(X)$ is closed [$\because$ f is continuous, X is compact
$\Rightarrow f(X)$ is compact
$\Rightarrow f(X)$ is closed]

$\Rightarrow M \in f(X)$

$\Rightarrow \exists\, x_0 \in X$ such that $M = f(x_0)$.

$m = \inf_{x \in X} f(x)$ is a limit point of $f(X)$

Then $m \in \overline{f(X)}$

also, $f(X)$ is closed

$\Rightarrow m \in f(X)$

$\Rightarrow \exists\, x_1 \in X$ st $m = f(x_1)$

$M = \max_{x \in X} f(x)$.

02/12/24 # Theorem: Let $f: X_1 \to Y$ be a continuous fn b/w two metric spaces X and Y. If $E \subseteq X$, is connected then $f(E)$ is also connected.

Proof: Assume that $f(E)$ is disconnected

$\Rightarrow f(E) = A \cup B$, where $A \neq \phi$, $B \neq \phi$
$A \cap \bar{B} = \phi$, $\bar{A} \cap B = \phi$

Let $f^{-1}(A) \cap E = G$

and $f^{-1}(B) \cap E = H$

then $G \cup H = (f^{-1}(A) \cap E) \cup (f^{-1}(B) \cap E)$

$= (f^{-1}(A) \cup f^{-1}(B)) \cap E \subseteq E$

$E \subseteq f^{-1}(f(E))$

$= f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow E \subseteq [f^{-1}(A) \cup f^{-1}(B)] \cap E$

$= (f^{-1}(A) \cap E) \cup (f^{-1}(B) \cap E)$

$= G \cup H$.

Hence $E = G \cup H$

as $G = f^{-1}(A) \cap E$ where $A \subseteq f(E)$

$\Rightarrow G \neq \phi$

Illy $H \neq \phi$

as $A \subseteq \bar{A}$

$\Rightarrow f^{-1}(A) \subseteq f^{-1}(\bar{A})$

$\Rightarrow G = f^{-1}(A) \cap E \subseteq f^{-1}(A) \subseteq f^{-1}(\bar{A})$

$\bar{G} \subseteq \overline{f^{-1}(\bar{A})}$

$= f^{-1}(\bar{A})$ [$\because f$ is continuous and $\bar{A}$ is closed]

also $H = f^{-1}(B) \cap E \subseteq f^{-1}(B)$

$\bar{G} \cap H \subseteq f^{-1}(\bar{A}) \cap f^{-1}(B)$

$= f^{-1}(\bar{A} \cap B)$

$= f^{-1}(\phi) = \phi$

Illy, $G \cap \bar{H} = \phi$

$\Rightarrow$ E is disconnected

$\rightarrow\leftarrow$

$\therefore$ E is connected.

Intermediate Value Theorem: Let f be a real valued continuous function on $[a,b]$. If $f(a) < f(b)$ and $c \in \mathbb{R}$ such that $f(a) < c < f(b)$, then $\exists\ x \in (a,b)$ such that $c = f(x)$

OR A continuous real function assumes all intermediate values on an interval.

Proof: $f : [a,b] \longrightarrow \mathbb{R}$

as $[a,b]$ is connected, and f is continuous

$\Rightarrow$ ~~f([a,b])~~ $f([a,b])$ is also connected in $\mathbb{R}$

$\Rightarrow f([a,b])$ is an interval in $\mathbb{R}$

$\Rightarrow f([a,b]) = [f(a), f(b)]$.

Let $c \in \mathbb{R}$ such that $c \in (f(a), f(b))$

Ho Sakti.

$\Rightarrow c \in f((a,b))$

$\Rightarrow \exists\ x \in (a,b)$ such that $c = f(x)$.

Monotonic functions $\rightarrow$ Increasing, strictly increasing, decreasing, strictly decreasing.

Theorem: Let f be a monotonically increasing function on (a,b). then $f(x^+)$ and $f(x^-)$ exist at every point $x \in (a,b)$. More precisely

$$\sup_{a<t<x} f(t) = f(x^-) \leq f(x) \leq f(x^+) = \inf_{x<t<b} f(t)$$

Furthermore if $a < x < y < b$, then $f(x^+) \leq f(y^-)$

Proof: $\{f(t) ; a < t < x\}$ is bounded above by $f(x)$

So it has a lower upper bound say 'A'.

i.e., $A = \sup_{a<t<x} f(t)$, also $A \leq f(x)$

for $\epsilon > 0$, $A - \epsilon$ is not lower upper bound

$\Rightarrow \exists\ \delta > 0$ st

$A - \epsilon < f(x - \delta) \leq A$ where $a < x - \delta < x$

as f is monotonically increasing, for $x-\delta < t < x$ then
$$A-\epsilon < f(x-\delta) \leq f(t) \leq A \leq f(x)$$

also, $A < A+\epsilon$

$\Rightarrow$ for $x-\delta < t < x$ then $A-\epsilon < f(t) < A+\epsilon$

$\Rightarrow |f(t) - A| < \epsilon$.

$\Rightarrow \lim\limits_{t \to x^-} f(t)$ exists and $= A$

$$f(x^-) = A$$

Ily, we can prove that $f(x^+)$ exists.

$$f(x^+) = \inf_{x<t<b} f(t)$$

as f is increasing, clearly

$$f(x^-) \leq f(x) \leq f(x^+)$$

~~Assume~~ Now, if $a < x < y < b$

$$f(x^+) = \inf_{x<t<b} f(t) \quad \inf_{x<t<y} f(t)$$

$S_1 = \{f(t);\ x < t < b\}$ $\quad S_2 \subseteq S_1$

$S_2 = \{f(t),\ x < t < y\}$

$$f(x^+) = \inf_{x<t<b} f(t) \leq \inf_{x<t<y} f(t)$$

$$f(y^-) = \sup_{a<t<y} f(t) \geq \sup_{x<t<y} f(t)$$

$$f(y^-) \geq \sup_{x<t<y} f(t) \geq \inf_{x<t<y} f(t) \geq f(x^+)$$

Hence proved

03/12/21 Theorem: Let f be monotonic on $[a,b]$. Then set of points at which f is discontinuous is atmost countable.

Proof: $f: [a,b] \to \mathbb{R}$

Let $E = \{c \in [a,b];\ f$ is discontinuous at $c\}$

Let $x \in E$

$\Rightarrow f(x^-) \neq f(x^+)$ (1) or $f(x^-) = f(x^+) \neq f(x)$ (2)

also $f(x^-) \leq f(x^+)$

and as f is increasing

$$\Rightarrow f(x^-) \leq f(x) \leq f(x^+)$$

If $f(x^-) = f(x^+) \Rightarrow f(x^-) = f(x) = f(x^+)$

$\Rightarrow$ Case 2 is not possible.

$\Rightarrow$ If f is discontinuous at $x \in E$ then

$f(x^-) \neq f(x^+)$

$\Rightarrow f(x^-) < f(x^+)$

for each $x \in E$, we associate a rational number $r(x)$ such that,

$f(x^-) < r(x) < f(x^+)$.

Let $x_1, x_2 \in E$ such that $x_1 < x_2$.

$\Rightarrow f(x_1^+) < f(x_2^-)$

$\Rightarrow f(x_1^-) < f(x_1^+) \leq f(x_2^-) < f(x_2^-)$

$\Rightarrow r(x_1) < r(x_2)$

$\Rightarrow$ If $x_1 \neq x_2$ then $r(x_1) \neq r(x_2)$] well defined

$g: E \to \mathbb{Q}$, $\boxed{g(x) = r(x)}$

Obviously $\Rightarrow$ g is one-one

$\boxed{E \text{ is countable}}$ ($\mathbb{Q}$ is countable)

06/12/21

Steiltjes.

Reimann Integration :→

Def^n: Let α be monotonically increasing function on $[a,b]$ and f be a bounded fun on $[a,b]$ corresponding to each partition $P: x_0 = a < x_1 < x_2 < \dots x_n = b$ of $[a,b]$

we define →

$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ for $i = 1, 2, \dots n$

Clearly, $\Delta \alpha_i \geq 0 \ \forall i$

We take $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$

$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$

$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i$ $\qquad$ $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$

$\overline{\int_a^b} f \, d\alpha = \inf_P U(P, f, \alpha)$, $\quad \underline{\int_a^b} f \, d\alpha = \sup_P L(P, f, \alpha)$

If $\overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha$ we say that f is reimann steiltjes integrable and we write $f \in R(\alpha)$ and the common value of 1 and 2 is denoted by

$\overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha = \int_a^b f \, d\alpha$

Def^n: Given a partition P of $[a,b]$ refinement of P is partition P^* of $[a,b]$ such that

$P \subseteq P^*$. If P_1 and P_2 are two partitions of $[a,b]$ then $P_1 \cup P_2$ is called common refinement of P_1 & P_2.

Theorem: If $P \subseteq P^*$ then

(i) $U(P^*, f, \alpha) \le U(P, f, \alpha)$

(ii) $L(P^*, f, \alpha) \geq L(P, f, \alpha)$

Proof: Let $P: a = x_0 < x_1 < \dots < x_n = b$

We consider the case when P^* has just one extra point than P, say x^*

$$P^* : a = x_0 < x_1 < \dots x_i < x^* < x_{i+1} < \dots < x_n = b$$

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$= M_1 \Delta \alpha_1 + M_2 \Delta \alpha_2 + \dots M_n \Delta \alpha_n$$

$$= M_1 \Delta \alpha_1 + M_2 \Delta \alpha_2 + \dots M_i \Delta \alpha_i + M_{i+1} \Delta \alpha_{i+1} + M_n \Delta \alpha_n$$

$$U(P^*, f, \alpha) = M_1 \Delta \alpha_1 + M_2 \Delta \alpha_2 + \dots + M_i \Delta \alpha_i + M_1^* (\alpha(x^*) - \alpha(x_i)) + M_2^* (\alpha(x_{i+1}) - \alpha(x^*)) + M_{i+2} \Delta \alpha_{i+2} + \dots M_n \Delta \alpha_n$$

where $M_1^* = \sup_{x \in [x_i, x^*]} f(x)$ $\quad M_2^* = \sup_{x \in [x^*, x_{i+1}]} f(x)$

Now, $U(P, f, \alpha) - U(P^*, f, \alpha) = M_{i+1} \Delta \alpha_{i+1} - M_1^* (\alpha(x^*) - \alpha(x_i)) - M_2^* (\alpha(x_{i+1}) - \alpha(x_i))$ —— (3)

As $M_{i+1} = \sup_{x \in [x_i, x_{i+1}]} f(x) \geq \sup_{x \in [x_i, x^*]} f(x) = M_1^*$

Similarly, $M_{i+1} = \sup_{x \in [x_i, x_{i+1}]} f(x) \geq \sup_{x \in [x^*, x_{i+1}]} f(x) = M_2^*$

So from (3), $U(P, f, \alpha) - U(P^*, f, \alpha) \geq M_{i+1} \Delta \alpha_{i+1} - M_{i+1} (\alpha(x^*) - \alpha(x_i)) - M_{i+1} (\alpha(x_{i+1}), \alpha(x^*))$

$$= M_{i+1} [\alpha(x_{i+1}) - \alpha(x_i) - \alpha(x^*) + \alpha(x_i) - \alpha(x_{i+1}) + \alpha(x^*)]$$

$$\boxed{U(P, f, \alpha) - U(P^*, f, \alpha) \geq 0}$$

Theorem: $\underline{\int_a^b} f \, d\alpha \le \overline{\int_a^b} f \, d\alpha$

Proof: Let P_1 and P_2 be two partitions of $[a,b]$.

Let $P^* = P_1 \cup P_2$ be common refinement of P_1 and P_2

$$L(P_1, f, \alpha) \le L(P^*, f, \alpha) \le U(P^*, f, \alpha) \le U(P_2, f, \alpha)$$

$\Rightarrow L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

If P_2 is fixed and sup is taken over all P_1,

$$\Rightarrow \underline{\int_a^b} f\, d\alpha \leq U(P_2, f, \alpha)$$

Now take infimum over all P_2

$$\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha$$

Theorem: $f \in R(\alpha)$ on $[a,b]$ iff given $\epsilon > 0$, $\exists$ a partition P of $[a,b]$ such that, $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

Proof: Suppose $f \in R(\alpha)$ and $\epsilon > 0$ be given

as $\int_a^b f\, d\alpha = \sup_P L(P, f, \alpha) = \inf_P U(P, f, \alpha)$

$\Rightarrow \exists$ partitions P_1 and P_2 such that

$$L(P_1, f, \alpha) > \int_a^b f\, d\alpha - \frac{\epsilon}{2}$$

$$U(P_2, f, \alpha) < \int_a^b f\, d\alpha + \frac{\epsilon}{2}$$

Let $P = P_1 \cup P_2$

$$L(P, f, \alpha) \geq L(P_1, f, \alpha) > \int_a^b f\, d\alpha - \frac{\epsilon}{2}$$

$$U(P, f, \alpha) \leq U(P_2, f, \alpha) < \int_a^b f\, d\alpha + \frac{\epsilon}{2}$$

$$\Rightarrow L(P, f, \alpha) > \int_a^b f\, d\alpha - \frac{\epsilon}{2} \quad \text{--- (1)}$$

$$\& \quad U(P, f, \alpha) < \int_a^b f\, d\alpha + \frac{\epsilon}{2} \quad \text{--- (2)}$$

from (1), $-L(P, f, \alpha) < -\int_a^b f\, d\alpha + \frac{\epsilon}{2}$ --- (3)

Add 2 and 3

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Conversely: Suppose for every $\epsilon > 0 \; \exists P$ of $[a,b]$ such that,

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$L(P, f, \alpha) \leq \underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha \leq U(P, f, \alpha)$$

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$0 \leq \overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f\, d\alpha \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$f(x) = \frac{1}{x}$ (Not uniform continuous)

$\Rightarrow 0 \le \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha < \epsilon$ for every $\epsilon > 0$

$\Rightarrow 0 \le \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha \le 0$

$\Rightarrow \overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$

$\Rightarrow f \in R(\alpha)$

07/12/21 **Theorem** → If f is a continuous function on $[a,b]$, then $f \in R(\alpha)$ on $[a,b]$

Proof → Let $\epsilon > 0$ be given as f is continuous on $[a,b]$ so f is uniformly continuous on $[a,b]$. Let $\eta < \frac{\epsilon}{\alpha(b) - \alpha(a) + 1}$

$\Rightarrow$ Given $\eta > 0$, $\exists\ \delta(\epsilon) > 0$ such that

whenever $|x - t| < \delta \Rightarrow |f(x) - f(t)| < \eta$

Let $P: a = x_0 < x_1 < \dots < x_n = b$ such that $\Delta x_i < \delta$

then $|M_i - m_i| < \eta$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$$

$$< \eta \sum_{i=1}^{n} \Delta\alpha_i$$

$$< \eta (\alpha(x_n) - \alpha(x_0)) \quad \left[\because \sum_{i=1}^{n} \Delta\alpha_i = \alpha(x_n) - \alpha(x_0)\right]$$

$$< \eta (\alpha(b) - \alpha(a))$$

$$< \frac{\epsilon}{\alpha(b) - \alpha(a) + 1} (\alpha(b) - \alpha(a))$$

$$< \epsilon$$

Theorem → If f is a monotonic function on $[a,b]$ and α is continuous on $[a,b]$ then $f \in R(\alpha)$

Proof → Let $\epsilon > 0$. Since α is continuous on $[a,b]$, it must assume all values between $\alpha(a)$ and $\alpha(b)$.

Also $\alpha(a) < \alpha(b)$

Claim → We can choose a partition P of $[a,b]$ such that

$$\underset{\boxed{\alpha(x_{i+1}) - \alpha(x_i)}}{\underset{||}{\Delta\alpha_i}} = \frac{\alpha(b) - \alpha(a)}{n}$$

Choose, $x_0 = a$ and x_1 such that

$$\alpha(x_1) - \alpha(x_0) = \frac{\alpha(b) - \alpha(a)}{n}$$

i.e., $\alpha(x_1) - \alpha(a) = \dfrac{\alpha(b) - \alpha(a)}{n}$

$$\alpha(x_1) = \alpha(a) + \frac{\alpha(b) - \alpha(a)}{n}$$

$\in [\alpha(a), \alpha(b)]$

and by intermediate value property, such x_1 exists

Choose, x_2 such that $\alpha(x_2) - \alpha(x_1) = \dfrac{\alpha(b) - \alpha(a)}{n}$

$$\alpha(x_2) = \alpha(x_1) + \frac{\alpha(b) - \alpha(a)}{n}$$

$$= \alpha(a) + \frac{2}{n}\left(\alpha(b) - \alpha(a)\right)$$

By IVP, we can choose such x_2.

we can keep on doing this process and find that -

$$\boxed{\alpha(x_n) = \alpha(b)}$$

$P. x_0 = a < x_1 < x_2 < - - - < x_n = b$

Now, $U(P, f, \alpha) - L(P, f, \alpha) =$

$$\sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$$

$$= \sum_{i=1}^{n} (M_i - m_i) \left(\frac{\alpha(b) - \alpha(a)}{n}\right)$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (M_i - m_i)$$

Monotonically increasing

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1})$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \left[f(x_n) - f(x_0)\right]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \left[f(b) - f(a)\right]$$

Choose n to be large enough so that $\dfrac{\alpha(b) - \alpha(a)}{n} (f(b) - f(a)) < \epsilon$

$\therefore f \in R(\alpha)$

Monotonically Decreasing

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} f(x_{i-1}) - f(x_i)$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_0) - f(x_n)$$

Same ——

Theorem → Suppose $f \in R(\alpha)$ on $[a,b]$, $m \le f \le M$, ϕ is a continuous fxn on $[m,M]$ and $h(x) = \phi(f(x))$, then $h \in R(\alpha)$ on $[a,b]$

Proof Let $\epsilon > 0$ be given

as ϕ is continuous on $[m,M]$

$\Rightarrow \phi$ is uniformly continuous on $[m,M]$

for given $\epsilon > 0$ $\exists\ \delta > 0$ such that

$$|s-t| < \delta \Rightarrow |\phi(s) - \phi(t)| < \epsilon$$

as f is riemann integrable

$\Rightarrow \exists$ partition $P = \{x_0, x_1, \ldots x_n\}$ of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon = \delta^2$$

$M_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$ $\qquad m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$

$M_i^* = \sup\limits_{x \in [x_{i-1}, x_i]} h(x)$ $\qquad m_i^* = \inf\limits_{x \in [x_{i-1}, x_i]} h(x)$

Divide $1, 2, \ldots n$ into 2 parts →

If $M_i - m_i < \delta$ then $i \in A$

If $M_i - m_i \geq \delta$ then $i \in B$

i.e., $\{1, 2, 3, \ldots n\} = A \cup B$ where $A \cap B = \phi$

If $i \in A$ then $f(x) \in [m_i, M_i]$ & $M_i - m_i < \delta$

So, by uniform continuity of ϕ

$$M_i^* - m_i^* < \epsilon$$

If $i \in B$ then If ϕ is bounded by K.

$$|M_i^* - m_i^*| \le |M_i^*| + |m_i^*|$$
$$\le K + K = 2K$$

$$< \sum_{i \in A} \epsilon\, \Delta\alpha_i + \sum_{i \in B} 2K\, \Delta\alpha_i \quad \text{---} (*)$$

Now, $\sum\limits_{i \in B} (M_i - m_i)\Delta\alpha_i$

$$\delta \sum_{i \in B} \Delta\alpha_i \le U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$$

$$\Rightarrow \sum_{i \in B} \Delta\alpha_i < \delta$$

Put this value in $(*)$ equation,

$$U(P,h,\alpha) - L(P,h,\alpha) < \epsilon \sum_{i \in A} \Delta\alpha_i + 2K\delta$$
$$< \epsilon(\alpha(b) - \alpha(a)) + 2K\delta$$

$< \epsilon \left(\alpha(b) - \alpha(a) + 2k \right)$ if $\delta > \epsilon$

$< \epsilon'$

$\Rightarrow h \in R(\alpha)$

Hence proved

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Theorem → If f and $g \in R(\alpha)$ on $[a,b]$ then

(a) $fg \in R(\alpha)$

(b) $|f| \in R(\alpha)$ and $\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha$

Proof

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2 \right]$$

Let $\phi(t) = t^2$

then ϕ is a continuous function

$(\phi \circ (f+g))(x) = \phi \circ (f(x) + g(x))$

$\Rightarrow \phi \circ (f+g) = (f(x) + g(x))^2$

$= (f+g)^2(x)$

$$\boxed{\phi \circ (f+g) = (f+g)^2}$$

(b) Define $\phi(t) = |t|$

then $(\phi \circ f)(x) = \phi(f(x))$

$= |f(x)|$

$= |f|(x)$

$\phi \circ f = |f|$

$\left| \int_a^b f \, d\alpha \right| = \pm \int_a^b f \, d\alpha$

$c \int_a^b f \, d\alpha$ where $c = \pm 1$

$= \int_a^b c f \, d\alpha \leq \int_a^b |f| \, d\alpha$

Hence proved

But converse is not true

For example → $|f| \in R(\alpha) \not\Rightarrow f \in R(\alpha)$

Let $\alpha(x) = x$, $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$

in $[0,1]$

Let $P: x_0 = 0 < x_1 < x_2 < \cdots < x_n = 1$ be partition of $[0,1]$

$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$

$= \sum_{i=1}^{n} (-1) \Delta\alpha_i$

$= -\sum_{i=1}^{n} \alpha(x_i) - \alpha(x_{i-1})$

$= -(\alpha(1) - \alpha(0))$

$= -(1-0)$
$= -1$

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$
$$= \sum_{i=1}^{n} (+1)\Delta\alpha_i$$
$$= \sum_{i=1}^{n} \alpha(x_i) - \alpha(x_{i-1})$$
$$= +(\alpha(1) - \alpha(0))$$
$$= +(1-0)$$
$$= 1$$

$$\overline{\int_a^b} f\,d\alpha = \inf_P U(P,f,\alpha) = 1$$

$$\underline{\int_a^b} f\,d\alpha = \sup_P L(P,f,\alpha) = -1$$

This fxn is not riemann steils Integrable

$\boxed{|f| = 1} \to$ constt fxn

Constt fxn is continuous and Continuous fxn is steilz Integrable but converse is not true.

Integration of Vector Valued functions

Defn Let $f_1, f_2, --- f_k$ be real functions on $[a,b]$ and let $f = (f_1, f_2, --- f_k) : [a,b] \to \mathbb{R}^k$

If α is increasing on $[a,b]$ we say $f \in R(\alpha)$ if $f_j \in R(\alpha)$ $\forall$ $1 \le j \le k$. ~~______~~ and

define $\int_a^b f\,d\alpha = \left(\int_a^b f_1\,d\alpha, \int_a^b f_2\,d\alpha, --- \int_a^b f_k\,d\alpha\right)$

Theorem If $f : [a,b] \to \mathbb{R}^k$, $f \in R(\alpha)$, where α is increasing on $[a,b]$ then $\|f\| \in R(\alpha)$ and

$$\left\| \int_a^b f\,d\alpha \right\| \le \int_a^b \|f\|\,d\alpha.$$

Proof Let $f = (f_1, f_2, ---- f_k)$, where

$f_i : [a,b] \to \mathbb{R}$ $\forall$ $1 \le i \le k$

$\|f\| = \sqrt{f_1^2 + f_2^2 + ---- f_k^2}$

$f \in R(\alpha)$, $f_i \in R(\alpha)$ $\forall$ $1 \le i \le k$

$\Rightarrow f_1^2 + f_2^2 + --- + f_k^2 \in R(\alpha)$

Define, $\phi(t) = \sqrt{t}$ (Continuous)

$\Rightarrow \phi \circ g \in R(\alpha) \Rightarrow \sqrt{f_1^2 + ---- f_k^2} \in R(\alpha)$

$\Rightarrow \boxed{\|f\| \in R(\alpha)}$

$$\int_a^b f\,d\alpha = \left(\int_a^b f_1\,d\alpha, \int_a^b f_2\,d\alpha, -- \int_a^b f_k\,d\alpha\right)$$

Cauchy Schwarz Inequality :-

$\cdot \|xy\| \leq \|x\|\|y\|$

Expanded form - $\sum_{i=1}^{k} x_i y_i \leq \sqrt{\sum_{i=1}^{k} x_i^2}\sqrt{\sum_{i=1}^{k} y_i^2}$

$= (y_1, y_2 - - - y_k)$ (say)

Now, $\left\|\int_a^b f\,d\alpha\right\|^2 = y_1^2 + y_2^2 + - - y_k^2$

$= \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{k} \left(y_i \int_a^b f_i\,d\alpha\right)$

$= \int_a^b \sum_{i=1}^{n} y_i f_i\,d\alpha$

$\leq \int_a^b \left(\sqrt{\sum_{i=1}^{k} y_i^2}\sqrt{\sum_{i=1}^{k} f_i^2}\right) d\alpha$

$= \sqrt{\sum_{i=1}^{k} y_i^2} \int \sqrt{\sum_{i=1}^{k} f_i^2}\,d\alpha$

$= \|y\| \int_a^b \|f\|\,d\alpha$

$\Rightarrow \|y\|^2 \leq \|y\| \int_a^b \|f\|\,d\alpha$ —— (1)

If $\|y\| = 0$

then $0 = y = \int_a^b \|f\|\,d\alpha$

If $\|y\| > 0$

then dividing (1) by $\|y\|$

$\|y\| \leq \int_a^b \|f\|\,d\alpha$

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UNIT - 03

Sequence and Series of Functions

Def: Let $\{f_n\}$ be a sequence of real valued functions defined on any set E. The sequence $\{f_n\}$ converges pointwise to a real valued function f defined on E, written as

$f_n \xrightarrow{P.W} f$ on E if

$\lim_{n\to\infty} f_n(x) = f(x) \;\forall\, x \in E$

i.e, for each $x \in E$, given $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that

$|f_n(x) - f(x)| < \epsilon \;\forall\, n \geq N$.

Example: let $f_n(x) = x^n$, $f_n : [0,1] \to \mathbb{R}$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n$

$= \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ 0 & \text{if } 0 < x < 1 \end{cases}$

$$= \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$f : [0,1] \to \mathbb{R}$

Def^n If $\Sigma f_n(x)$ converges for every $x \in E$ (pointwise) and if we define

$f(x) = \sum_{n=1}^{\infty} f_n(x)$, $x \in E$ then the fxn f is called the sum of the series Σf_n

Example

$f_n(x) = x^n$ (not continuous at $x=1$)

$$\begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$f_n(x) \to f(x)$

$\lim_{n\to\infty} f_n(x) = f(x)$

$\int \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int f_n(x)$

Example $f_n : [0,1] \to \mathbb{R}$

$f_n(x) = n^2 x (1-x^2)^n$

Sol^n $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n^2 x (1-x^2)^n$

$\lim_{n\to\infty} n^2 x \frac{1}{(1-x^2)^{-n}} = 0$

$\int_0^1 \lim_{n\to\infty} f_n(x) = \int_0^1 0\, dx = 0$

$f_1(x) = x(1-x^2)$

$\int_0^1 x(1-x^2)^n dx = \int_0^1 \frac{(1-t)^n}{2} dt$

$= \cancel{\left[\frac{1t}{2} - \frac{t^2}{4}\right]_0^1 = \frac{1}{2}}$

$= \frac{(1-t)^{n+1}}{2(n+1)} (-1) \Big|_0^1$

$= \frac{1}{2n+2}$

$\int f_n(x) = n^2 \int x(1-x^2)^n dx$

$= \frac{n^2}{2n+2}$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2}{2n+2} = \infty$.

Def^n The sequence $\{f_n\}$ is said to converge uniformly to a real valued function f on E if for any $\epsilon > 0$ $\exists$ a +ve integer

N (independent of x, but dependent on ϵ) st $\forall$ $x \in E$,
$|f_n(x) - f(x)| < \epsilon$ $\forall$ $n \geq N$.

NOTE→ It is clear that every uniformly convergent sequence of functions is pointwise convergent.

NOTE→ Converse is not always true. (Pointwise convergent not be uniform convergent)

Cauchy's Criterion for Uniform Convergence →

Statement→ The sequence of functions $\{f_n\}$ defined on E, converges uniformly on E iff given $\epsilon > 0$ $\exists$ $N(\epsilon) \in \mathbb{N}$ such that, $|f_n(x) - f_m(x)| < \epsilon$ $\forall$ $n, m \geq N$ — (1).

Proof→ Let $\{f_n\}$ converge uniformly to f.

then given $\epsilon > 0$ $\exists$ $N(\epsilon) \in \mathbb{N}$ such that

$|f_n(x) - f(x)| < \frac{\epsilon}{2}$ $\forall$ $n \geq N$

Now, let $m, n \geq N$

then $|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)|$

$\leq |f_n(x) - f(x)| + |f_m(x) - f(x)|$
$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Conversely→ If (1) satisfied then $f_n(x) \to f(x)$ we just need to show that the convergence is uniform for a given $\epsilon > 0$ choose integer N such that

$|f_n(x) - f_m(x)| < \epsilon$ $\forall$ $n, m \geq N$

fix 'n' and take $m \to \infty$

then, $|f_n(x) - f(x)| < \epsilon$ $\forall$ $n \geq N$ where n depends only on ϵ.

→ $f_n(x) - f(x)$ is uniformly converges

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M_n test for uniform convergence of sequence of functions →

Statement→ Let $\{f_n\}$ be a sequence of functions such that $\lim_{n \to \infty} f_n(x) = f(x)$ $x \in [a,b] = E$ and let

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|$$

then $f_n \to f$ uniformly on $[a,b]$ iff $M_n \to 0$ as $n \to \infty$.

Proof→ Let $f_n \to f$ uniformly on $[a,b]$

So, for given $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that

$|f_n(x) - f(x)| < \epsilon$ $\forall$ $n \geq N$

$\Rightarrow M_n = \sup_{x \in E} |f_n(x) - f(x)|$

$\Rightarrow M_n < \epsilon \ \forall\ n \geq N$

$\Rightarrow |M_n - 0| < \epsilon \ \forall\ n \geq N \Rightarrow M_n \to 0$ as $n \to \infty$

Conversely: Now let $M_n \to 0$ as $n \to \infty$

$\Rightarrow$ given $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ st

$|M_n - 0| < \epsilon \ \forall\ n \geq N$

$\Rightarrow M_n < \epsilon \ \forall\ n \geq N$

$\Rightarrow \sup_{x \in E} |f_n(x) - f(x)| < \epsilon \ \forall\ n \geq N$

$\Rightarrow |f_n(x) - f(x)| < \epsilon \ \forall\ n \geq N, \forall\ x \in E$

$\Rightarrow f_n \to f$ uniformly.

Ques: $f_n(x) = n^2 x (1 - x^2)^n$, $x \in [0, 1]$.

Sol: $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} n^2 x (1 - x^2)^n = 0$

$\Rightarrow f_n(x) \to f(x)$ pointwise, where $f(x) = 0$ $\forall\ x \in [0, 1]$.

$M_n = \sup_{x \in E} |f_n(x) - f(x)|$

$= \sup_{x \in [0,1]} |n^2 x (1 - x^2)^n|$

$g(x) = n^2 x (1 - x^2)^n$

$g'(x) = n^2 (1 - x^2)^n + n^2 x n (1 - x^2)^{n-1} (-2x)$

$g'(x) = 0$

$\Rightarrow n^2 (1 - x^2)^{n-1} [(1 - x^2)^n + (-2x^2) n] = 0$

$\Rightarrow n^2 (1 - x^2)^{n-1} [(1 - x^2)^n - 2x^2 n] = 0$

$\Rightarrow n^2 (1 - x^2)^{n-1} [1 - (1 + 2n) x^2] = 0$

$\Rightarrow (1 - x^2) = 0$ OR $1 - (1 + 2n) x^2 = 0$

$x = \pm 1$ $\qquad x = \pm \frac{1}{\sqrt{1+2n}}$

$g''(x) =$

at $x = \frac{1}{\sqrt{1+2n}}$, $g''(x) = < 0$

$\Rightarrow \frac{1}{\sqrt{1+2n}}$ is a pt of maxima

$M_n = \sup_{x \in E} |n^2 x (1 - x^2)^n|$

$\Rightarrow M_n = \frac{n^2}{\sqrt{1+2n}} \left(1 - \frac{1}{1+2n}\right)^n$

$\lim\limits_{n\to\infty} M_n = \lim\limits_{n\to\infty} \frac{n^2}{\sqrt{1+2n}} \left(1 - \frac{1}{1+2n}\right)^n$

$= \frac{\infty}{1}\left(1 - \frac{1}{\infty}\right)^{\infty} = \infty \cdot (0) \neq 0.$

$= \lim\limits_{n\to\infty} \frac{n^{3/2}}{\sqrt{\frac{1}{n}+2}} (1)$

$= \infty$

Cauchy's Criterion for Uniform convergence of series →

Statement → A series $\sum_{n=1}^{\infty} u_n$ of real valued functions defined on E, converges uniformly on E iff for given $\epsilon > 0$, $\exists$ $N(\epsilon) \in N$

$$|u_{n+1}(x) + u_{n+2}(x) + \cdots u_{n+p}(x)| < \epsilon \quad \forall p = 1, 2, \ldots$$

Proof: Let $S_n = u_1 + u_2 + \cdots u_n$

then the series $\sum u_n$ converges uniformly on E iff $\{S_n\}$ converges uniformly on E

$\Rightarrow$ By Cauchy Criteria for Uniform convergence of sequence of functions-

given $\epsilon > 0$, $\exists N(\epsilon) \in N$ st,

$$|S_n - S_m| < \epsilon \quad \forall n, m \geq N \quad \forall x \in E$$

Let $n < m \Rightarrow m = n + p$ for some $p \in N$

$$|S_n(x) - (S_n(x) + u_{n+1}(x) + \cdots u_{n+p}(x))| < \epsilon \quad \forall p \in N, x \in E$$

$$\Rightarrow |u_{n+1}(x) + u_{n+2}(x) + \cdots u_{n+p}(x)| < \epsilon \quad \forall p \in N, \forall x \in E$$

Weirstrass M-test →

Statement → Suppose $\{f_n(x)\}$ is a sequence of functions defined on E and

$|f_n(x)| \leq M_n \quad \forall n$ and $\forall x \in E$.

then, $\sum f_n(x)$ converges uniformly if $\sum M_n$ converges.

Proof: Assume $\sum M_n$ converges

$\Rightarrow$ for given $\epsilon > 0$ $\exists$ $m \in N$ such that

$$M_{n+1} + M_{n+2} + \cdots + M_{n+p} < \epsilon \quad \forall n \geq m, \forall p \in N$$

$|S_n - S_m| < \epsilon$ ($m > n$)

also $|f_n(x)| \le M_n \ \forall n \ \& \ \forall x \in E$.

$$|f_{n+1}(x) + f_{n+2}(x) + \dots f_{n+p}(x)| \le |f_{n+1}(x)| + |f_{n+2}(x)| + \dots + |f_{n+p}(x)|$$

$$\le M_{n+1} + M_{n+2} + \dots M_{n+p}$$
$$< \epsilon \ \forall n \ge M, \ \forall p \in N.$$

$\Rightarrow \sum f_n(x)$ converges uniformly.

13/12/21 **Uniform Convergence and Continuity** →

Suppose $f_n \to f$ uniformly on a set E in a metric space X. Let $x \in E'$. Suppose $\lim_{t \to x} f_n(t) = A_n$. Then $\{A_n\}$ converges and $\lim_{n\to\infty} A_n = \lim_{t\to x} \lim_{n\to\infty} f_n(t)$

i.e.,

$$\lim_{n\to\infty} \lim_{t\to x} f_n(t) = \lim_{t\to x} \lim_{n\to\infty} f_n(t).$$

Proof: As $f_n \to f$ uniformly, so for given $\epsilon > 0$

$\exists N(\epsilon)$ st $\forall n, m \ge N$

$$|f_n(t) - f_m(t)| < \frac{\epsilon}{3} \quad \forall t \in E \quad \text{— (1)}$$

We take $t \to x$

$$|A_n - A_m| < \frac{\epsilon}{3} \quad \forall n, m \ge N \quad \text{— (2)}$$

$\{A_n\}$ is a sequence of real numbers, which is also a cauchy sequence

as R is complete metric space, $\{A_n\}$ converges to some real number.

Let $\lim_{n\to\infty} A_n = A$.

Now, $|f(t) - A| = |f(t) - f_N(t) + f_N(t) - A_N + A_N - A|$

$$\le |f(t) - f_N(t)| + |f_N(t) - A_N| + |A_N - A| \quad \text{— (3)}$$

from 1 and 2, putting $m = N$ and taking $n \to \infty$, we get

$$|f(t) - f_N(t)| < \frac{\epsilon}{3} \text{ and}$$
$$|A - A_N| < \frac{\epsilon}{3}$$

$\lim_{t\to x} f_N(t) = A_N \Rightarrow \exists$ a nbd V of x st

$$|f_N(t) - A_N| < \frac{\epsilon}{3} \quad \forall t \in V, t \ne x$$

from (3), $|f(t) - A| < \epsilon \quad \forall t \in V, t \ne x$

$$\Rightarrow \lim_{t\to x} f(t) = A.$$

$$\Rightarrow \lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$$

Hence proved

Corollary → If $\{f_n\}$ is a sequence of continuous fns on E and $f_n \to f$ uniformly on E, then f is continuous on E.

Proof: $A_n = \lim_{t \to x} f_n(t)$

$= f_n(x)$ [by continuity of f_n]

Also, $\lim_{n\to\infty} A_n = \lim_{t\to x} \lim_{n\to\infty} f_n(t)$

$\Rightarrow \lim_{n\to\infty} f_n(x) = \lim_{t\to x} f(t)$

$f(x) = \lim_{t\to x} f(t)$

$\Rightarrow f$ is continuous at x.

- **Uniform Convergence and Integration** →

Theorem: Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in R(\alpha)$ on $[a,b]$ $\forall$ $n = 1, 2, \ldots$ and $f_n \to f$ uniformly on $[a,b]$. Then, $f \in R(\alpha)$ on $[a,b]$ and $\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha$

Proof: Let $\epsilon_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$

$\Rightarrow |f_n(x) - f(x)| \le \epsilon_n \quad \forall\, x \in [a,b]$

$\Rightarrow -\epsilon_n \le f_n(x) - f(x) \le \epsilon_n \;\forall\, x \in [a,b]$

$\Rightarrow -\epsilon_n \le f(x) - f_n(x) \le \epsilon_n \quad \forall\, x \in [a,b]$

$\Rightarrow f_n(x) - \epsilon_n \le f(x) \le \epsilon_n + f_n(x) \quad \forall\, x \in [a,b]$

$\Rightarrow \int_a^b (f_n(x) \pm \epsilon_n)\,d\alpha = \int_a^b f_n\,d\alpha \pm \epsilon_n \int_a^b d\alpha$

$= \int_a^b f_n\,d\alpha \pm \epsilon_n(\alpha(b) - \alpha(a))$ — (*)

Also, $\int_a^b (f_n(x) - \epsilon_n)\,d\alpha \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$

$\le \int_a^b (f_n(x) + \epsilon_n)\,d\alpha$

$\Rightarrow$ ~~$\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha$~~

$\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le \int_a^b (f_n(x) + \epsilon_n)\,d\alpha - \int_a^b (f_n(x) - \epsilon_n)\,d\alpha$

$= \int_a^b f_n\,d\alpha + \epsilon_n(\alpha(b) - \alpha(a)) - \left(\int_a^b f_n\,d\alpha - \epsilon_n(\alpha(b) - \alpha(a))\right)$ From (*)

$= 2\epsilon_n(\alpha(b) - \alpha(a))$

$\epsilon_n = \sup_{x \in [a,b]} |f_n(x) - f(x)|$

$\epsilon_n \to 0$ as $n \to \infty$

$\Rightarrow$ as $n \to \infty$ $\int_a^b f\,d\alpha = \int_a^{\bar{b}} f\,d\alpha$

$\Rightarrow f \in R(\alpha)$ on $[a,b]$

Consider, $\left|\int_a^b f_n\,d\alpha - \int_a^b f\,d\alpha\right| =$

$\left|\int_a^b (f_n - f)\,d\alpha\right|$

$\leq \int_a^b |f_n - f|\,d\alpha \leq \int_a^b \epsilon_n\,d\alpha$

$= \epsilon_n(\alpha(b) - \alpha(a)) \to 0$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} \int_a^b f_n\,d\alpha = \int_a^b f\,d\alpha$

Corollary: If $f_n \in R(\alpha)$ on $[a,b]$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $a < x < b$ and convergence is uniform on $[a,b]$. Then

$\sum_{n=1}^{\infty} \int_a^b f_n\,d\alpha = \int_a^b f\,d\alpha$

Proof: Let $s_n(x) = \sum_{i=1}^{\infty} f_i(x) \in R(\alpha)$

$\Rightarrow s_n(x)$ is a sequence of Riemann sleiltjes Integration fxns on $[a,b]$ which converges to f uniformly.

$\Rightarrow f \in R(\alpha)$ on $[a,b]$ (by previous theorem)

and $\lim_{n\to\infty} \int_a^b s_n\,d\alpha = \int_a^b f\,d\alpha$

$\Rightarrow \lim_{n\to\infty} \int_a^b \sum_{i=1}^{n} f_i\,d\alpha = \int_a^b f.$

$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \int_a^b f_i\,d\alpha = \int_a^b f\,da$

$= \sum_{i=1}^{\infty} \int_a^b f_i\,d\alpha = \int_a^b f\,d\alpha$

or $\sum_{i=1}^{n} \int_a^b f_i\,d\alpha = \int_a^b f\,d\alpha$

14/12/21

Theorem: Suppose $\{f_n\}$ is a sequence of functions differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$ then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and $f'(x) = \lim_{n\to\infty} f_n'(x)$, $a \le x \le b$.

Proof: $\{f_n(x_0)\}$ is convergent. So given $\epsilon > 0$, $\exists\, N_1 \in \mathbb{N}$ st

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \quad \forall\, n,m \ge N_1$$

as $\{f_n'\}$ is uniformly cgt, $\exists\, N_2 \in \mathbb{N}$ such that

$$|f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)} \quad \forall\, n,m \ge N_2$$
and $\forall\, t \in [a,b]$.

Let $N = \max\{N_1, N_2\}$

$$\Rightarrow |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \quad \forall\, n,m \ge N$$

and $$|f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)} \quad \forall\, n,m \ge N$$

Apply mean value theorem, on $\{f_n - f_m\}$ on $[x,t]$ or $[t,x]$ we get, that $\forall\, m,n \ge N$

$$\frac{|f_n(x) - f_m(x) - (f_n(t) - f_m(t))|}{|t-x|} = |f_n'(\xi) - f_m'(\xi)|$$

where $\xi \in (x,t)$ or (t,x)

$$\Rightarrow |f_n(x) - f_m(x) - f_n(t) + f_m(t)| = |f_n'(\xi) - f_m'(\xi)|\,|t-x|$$

$$< \frac{\epsilon}{2(b-a)}\,|t-x|$$

$$< \frac{\epsilon}{2(b-a)}(b-a) = \frac{\epsilon}{2}.$$

In particular if $t = x_0$

$$|f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| < \frac{\epsilon}{2}$$

$$|f_n(x) - f_m(x)| = |f_n(x) - f_m(x) - f_n(x_0) + f_n(x_0) - f_m(x_0) + f_m(x_0)|$$

$$\le |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall\, n, m \geq N$

$\Rightarrow |f_n(x) - f_m(x)| < \epsilon \quad \forall\, n, m \geq N$

$\Rightarrow \{f_n\}$ is uniformly convergent.

Conversely let $\lim_{n \to \infty} f_n(x) = f(x)$

Suppose $x \in [a, b]$ be a fixed point

Define $\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \; t \neq x$

$\lim_{n \to \infty} \phi(t) = \frac{f(t) - f(x)}{t - x}, \; t \neq x.$

$\lim_{t \to x} \phi_n(t) = f_n'(x)$

$$|\phi_n(t) - \phi_m(t)| = \frac{|f_n(t) - f_n(x) - f_m(t) + f_m(x)|}{t - x}$$

$$< \frac{\epsilon}{2(b-a)}$$

$\{\phi_n\}$ is uniformly convergent

By theorem,

$$\lim_{n \to \infty} \lim_{t \to x} \phi_n(t) = \lim_{t \to x} \lim_{n \to \infty} \phi_n(t)$$

$$\lim_{n \to \infty} f_n'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

$$= f'(x)$$

Pointwise and uniform bounded function

Def^n: Let $\{f_n\}$ be a sequence of functions defined on E we say $\{f_n\}$ is pointwise bounded if $\{f_n(x)\}$ is bounded for each $x \in X$ i.e. $\exists$ a finite valued function ϕ such that –

$$|f_n(x)| \leq \phi(x) \quad \forall\, x \in E \quad \forall\, n = 1, 2, \ldots$$

We say $\{f_n\}$ is uniformly bounded if $\exists$ M such that

$$|f_n(x)| \leq M \quad \forall\, x \in E, \; n = 1, 2, \ldots$$

Equicontinuous functions $\rightarrow$ A family F of complex valued functions defined on set E in a metric space X is said to equicontinuous on E if for given $\epsilon > 0 \; \exists \; \delta > 0$ st

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } |x - y| < \delta$$

$\forall\, f \in F$.

$\forall\, x, y \in E$.

Note: Every member of equicontinuous family of functions is uniformly continuous!

Example Let $f_n : [0,1] \to \mathbb{R}$ such that

$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}$$

Sol: Let $x = \frac{1}{n}$, $y = \frac{1}{n+1}$

$$|x-y| = \left|\frac{1}{n} - \frac{1}{n+1}\right|$$

$$= \frac{1}{n(n+1)}$$

Now, $|f_n(x) - f_n(y)| =$

$$= \left|f_n\left(\tfrac{1}{n}\right) - f_n\left(\tfrac{1}{n+1}\right)\right|$$

$$f_n\left(\tfrac{1}{n}\right) = \frac{x^2}{x^2 + (1-nx)^2}$$

$$= \frac{1}{1 + \left(\frac{1}{x} - n\right)^2}$$

$$f_n\left(\frac{1}{n+1}\right) = \frac{1}{1 + (n+1-n)^2} = \frac{1}{1+1} = \frac{1}{2}$$

$$= \left|1 - \frac{1}{2}\right| = \frac{1}{2}$$

$$\Rightarrow \left|f_n\left(\tfrac{1}{n}\right) - f_n\left(\tfrac{1}{n+1}\right)\right| = \frac{1}{2} \ \forall n \in \mathbb{N}$$

Choose, $\epsilon = \frac{1}{2}$ then $\left|f_n\left(\frac{1}{n}\right) - f_n\left(\frac{1}{n+1}\right)\right| \not< \epsilon$

but $\left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \delta$

$\{f_n\}$ not a family of equicontinuous fxns.

16/12/21 **Imp.** **Weierstrass Theorem:** Let f be continuous complex function on $[a,b]$. Then $\exists$ a sequence of polynomials P_n s.t. $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a,b]$. If f is real, then the sequence P_n may be taken as reals.

Proof: Without loss of generality, we can take $[a,b] = [0,1]$.

$\because$ we can change the interval $[a,b]$ to $[0,1]$ by defining a map as $\to$

$$g(t) = f(a + t(b-a)); \ 0 \le t \le 1$$

equivalently $g\left(\frac{y-a}{b-a}\right) = f(y)$ $a \le y \le b$

and f is continuous on $[a,b]$ iff g is continuous on $[0,1]$.

We can also assume that $f(0) = f(1) = 0$

$\therefore$ If we consider $g(x) = f(x) - f(0) - x[f(1) - f(0)]$

then $g(0) = 0$, $g(1) = 0$

If Weirstrass theorem is proved for g, then $\exists$ a sequence of polynomials $P_n(x)$ such that $\lim_{n\to\infty} P_n(x) = g(x)$.

as $(f-g)(x) = f(0) + x(f(1) - f(0))$
constant

Thus the sequence of polynomials

$P_n^*(x) = P_n(x) + f(0) + x(f(1) - f(0))$

$\lim_{n\to\infty} P_n^*(x) = g(x) + (f-g)(x) = f(x).$

Define $f(x) = 0$ on $\mathbb{R} \setminus [0,1]$ then f is continuous on $\mathbb{R}$.

N Define $Q_n(x) = C_n(1-x^2)^n$, $n = 1,2,3..$

where C_n is chosen so that $\int_{-1}^{1} Q_n(x) = 1 \ \forall n$

$\Rightarrow \int_{-1}^{1} C_n(1-x^2)^n dx = 1$

$2\int_0^1 C_n(1-x^2)^n dx$

$\geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n dx$ for $n > 1$

$> 2C_n \int_0^{1/\sqrt{n}} (1-nx^2) dx$

$[\because (1-x^2)^n \geq 1 - nx^2]$

To prove - take $F(x) = (1-x^2)^n - 1 + nx^2$

then $F(0) = 0$

and $F'(x) > 0 \Rightarrow F$ is increasing

$\Rightarrow$ if $x > 0 \Rightarrow F(x) > F(0)$

$\Rightarrow (1-x^2)^n - 1 + nx^2 > 0$

$\Rightarrow (1-x^2)^n > 1 - nx^2$

$= 2C_n\left[x - \frac{nx^3}{3}\right]_0^{1/\sqrt{n}}$

$= 2C_n\left[\frac{1}{\sqrt{n}} - \frac{1}{3\sqrt{n}}\right] = 2C_n\left(\frac{2}{3\sqrt{n}}\right)$

$= \frac{4}{3} \frac{C_n}{\sqrt{n}}$

$1 > \frac{4}{3} \frac{C_n}{\sqrt{n}}$

$C_n < \frac{3\sqrt{n}}{4} < \sqrt{n}$

$\Rightarrow C_n < \sqrt{n}$

For any $\delta > 0$ and $\delta \le |x| \le 1$ then,

$\phi_n(x) = C_n(1-x^2)^n < \sqrt{n}\,(1-\delta^2)^n \to 0$ as $n \to \infty$

$\Rightarrow \phi_n(x)$ converges uniformly to 0 on $\delta \le |x| \le 1$.

Let $P_n(x) = \int_{-1}^{1} f(x+t)\,\phi_n(t)\,dt,\ 0 \le x \le 1$

$\Rightarrow \underbrace{\int_{-1}^{-x} f(x+t)\,\phi_n(t)\,dt}_{I_1} + \underbrace{\int_{-x}^{1-x}}_{I_2} + \underbrace{\int_{1-x}^{1} I_3}_{I_3}$

$I_3 = \int_{1-x}^{1} f(x+t)\,\phi_n(t)\,dt$

$I_1 \quad -1 \le t \le -x$

$-1+x \le t+x \le 0$

$t+x \in [-1+x, 0]$

$\boxed{f(t+x) = 0}$

$\Rightarrow \boxed{I_1 = 0}$

for $I_3 = 1-x \le t \le 1$

$1 \le t+x \le 1+x$

$t+x \in [1, 1+x]$

$\Rightarrow f(t+x) = 0$

$\Rightarrow \boxed{I_3 = 0}$

$\Rightarrow P_n(x) = \int_{-x}^{1-x} f(x+t)\,\phi_n(t)\,dt$

Put $x+t = y$

$dt = dy$

$P_n(x) = \int_0^1 f(y)\,\phi_n(y-x)\,dy$

$\boxed{P_n(x) = C_n \int_0^1 f(y)\,(1-(y-x)^2)^n\,dy}$

Since f is uniformly continuous on $[0,1]$

Given $\epsilon > 0$ $\exists$ $\delta > 0$ such that,

$|y-x| < \delta \Rightarrow |f(y) - f(x)| < \frac{\epsilon}{2}$

Let $M = \sup_{x \in [0,1]} |f(x)|$

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt - f(x)\int_{-1}^{1} Q_n(t)\, dt \right|$$

$$= \left| \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt - \int_{-1}^{1} Q_n(t) f(x)\, dt \right|$$

$$= \left| \int_{-1}^{1} \big(f(x+t) - f(x)\big)\, Q_n(t)\, dt \right|$$

$$\leq \int_{-1}^{1} |f(x+t) - f(x)|\, |Q_n(t)|\, dt$$

$$\int_{-1}^{-\delta} |f(x+t) - f(x)|\, Q_n(t)\, dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)|\, |Q_n(t)|\, dt$$

$$+ \int_{\delta}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt$$

$$\leq 2M \int_{-1}^{-\delta} Q_n(t)\, dt + \frac{\epsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\, dt + 2M\int_{\delta}^{1} Q_n(t)\, dt$$

$$= -2M \int_{1}^{\delta} Q_n(t)\, dt + \frac{\epsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\, dt + 2M\int_{\delta}^{1} Q_n(t)\, dt$$

$$= 2M \int_{\delta}^{1} Q_n(t)\, dt + \frac{\epsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\, dt + 2M\int_{\delta}^{1} Q_n(t)\, dt$$

$$= 4M \int_{\delta}^{1} Q_n(t)\, dt + \frac{\epsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\, dt$$

$$< 4M \sqrt{n}\,(1-\delta^2)^n + \frac{\epsilon}{2}$$

$$\longrightarrow 0 \text{ as } n \to \infty$$

Also $P_n(x) = C_n \int_0^1 f(y)\, [1-(y-x)^2]^n\, dy$ which is real polynomial of deg $\leq 2n+1$ if f is real valued fxn.

Hence if f is real valued, $P_n(x)$ is a sequence of real fxn.

27/10/21 **Corollary →** For every interval $[a,a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-a,a]$.

Proof: By weirstrass theorem, $\exists$ a sequence P_n^* of real polynomials such that $P_n^* \longrightarrow |x|$ uniformly on $[-a,a]$.

In particular, $P_n^*(0) \longrightarrow 0$ as $n \to \infty$.

Define $P_n(x) = P_n^*(x) - P_n^*(0)$

then $P_n(0) = P_n^*(0) - P_n^*(0) = 0$
as $n \to \infty$, $P_n(x) \to |x| - 0$

$\Rightarrow P_n(x) \longrightarrow |x|$ uniformly.

Definition: A family A of real / complex functions defined on E is called an algebra if

(i) $f+g \in A$
(ii) $fg \in A$
(iii) $cf \in A$
for all $f, g \in A$, $c \in \mathbb{R}$ or $\mathbb{C}$.

For example → (1) Set of all polynomials
$f: \mathbb{R} \to \mathbb{R}$
$f(x) = 2+3x$
(2) set of all complex polynomials.

Definition: An algebra A is said to be uniformly closed if whenever $\{f_n\} \in A$, $f_n \to f$ uniformly on E, then $f \in A$.

Definition: An algebra B is said to be uniform closure of algebra A if whenever $\{f_n\} \in B$ such that $f_n \to f$ uniformly, then $f \in B$.

Definition: Let A be a family of functions defined on a set E. A is said to separate points on E if for every $x_1, x_2 \in E$, $x_1 \neq x_2$, $\exists$ $f \in A$ such that $f(x_1) \neq f(x_2)$.

Definition: If $\forall x \in E$, $\exists g \in A$ st $g(x) \neq 0$ we say that A vanishes at no pt of E or A doesnot vanish at any pt of E.

Example: A = Algebra of all polynomials in one variable
$[1,2]$
let $x = 1.5$
$g(x) = x$

Theorem: Let A be an algebra of functions on a set E. A separate points on E and A vanishes at no point of E. Suppose x_1, x_2 are distinct pts of E and c_1, c_2 are constants. Then $\exists f \in A$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$

Proof: $\exists g \in A$ such that

$g(x_1) \neq g(x_2)$ ($\because$ A separate points)

and $\exists h, k \in A$ such that

$h(x_1) \neq 0$, $k(x_2) \neq 0$ ($\because$ A doesnot vanish on E)

Put $u = gk - g(x_1)k$
$v = gh - g(x_2)h$
$u, v \in A$

$u(x_1) = 0 \quad v(x_2) = 0$
$u(x_2) = (g(x_2) - g(x_1))\, k(x_2) \neq 0$
$v(x_1) = (g(x_1) - g(x_2))\, h(x_1) \neq 0$

take $f = \dfrac{c_1 v}{v(x_1)} + \dfrac{c_2 u}{u(x_2)} \in A$

$f(x_1) = c_1$
$f(x_2) = c_2$

Stone Weirstrass Theorem :-

Statement:- Let A be an algebra of all real continuous functions on a compact set K. If A separates pts on K and vanishes at no pt. of K, then the uniform closure $\bar{B}$ of A is A itself.

Proof (1) Claim:- $f \in B$ then $|f| \in B$.

Let $a = \sup_{x \in K} |f(x)|$ as f is bounded
$\Rightarrow a$ is finite

Let $\epsilon > 0$ be given

By corollary to Weirstrass theorem, $\exists$ real constants $c_1, c_2, \dots c_n$ such that

$$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \epsilon \text{ for } y \in [-a, a] \quad (*)$$

Consider $g = \sum_{i=1}^{n} c_i f^i$

then $g \in B$ [$\because B$ is an algebra]
[uniform closure of an algebra is again an algebra]

Let $y = f(x) \in [-a, a]$ in $(*)$

$\Rightarrow \left| \sum_{i=1}^{n} c_i y^i - |f(x)| \right| < \epsilon \ \forall f(x) \in [-a, a]$

$\Rightarrow |g(x) - |f(x)|| < \epsilon \ \forall x \in K$

$\Rightarrow B(|f(x)|, \epsilon) \cap B \neq \phi$

$\Rightarrow |f| \in \bar{B} = B$

Step 2- **Claim**- If $f, g \in B$ then
$\max\{f, g\} \in B$ & $\min\{f, g\} \in B$

$\max\{f, g\} = \dfrac{|f+g|}{2} + \dfrac{|f-g|}{2}$

$\min\{f, g\} = \dfrac{f+g}{2} - \dfrac{|f-g|}{2}$

as $f, g \in B \Rightarrow f+g \in B \Rightarrow \dfrac{f+g}{2} \in B$
$f - g \in B$

$\Rightarrow |f-g| \in B$ (by Step 1)
$\Rightarrow \dfrac{|f-g|}{2} \in B$

Similarly max and min of a finite no. of functions in B is again in B.

Step 3: Claim: Given a real function f, continuous on K, for a point $x \in K$ and $\epsilon > 0$ $\exists$ $g_x \in B$ such that

$g_x(x) = f(x)$ and $g_x(t) > f(t) - \epsilon \ \forall t \in K$

$\because A \subseteq B$, so B also separate points and doesnot vanish on K.

hence, for every $y \in K$, $c_1 = f(x)$, $c_2 = f(y)$

$\exists h_y \in B$ such that $h_y(x) = c_1 = f(x)$
$h_y(y) = c_2 = f(y)$
[By previous theorem]

Since h_y is continuous function and $(h_y - f)_y = 0$

$\Rightarrow$ Given $\epsilon > 0$ $\exists$ an open set $J_y \ni y$ such that

$-\epsilon < (h_y - f)(t) < \epsilon$

$K \subseteq \bigcup_{y \in K} J_y$

as K is compact, so $\exists$ $y_1, y_2, \ldots y_n$ st,

$K \subseteq J_{y_1} \cup J_{y_2} \cup \ldots \ldots J_{y_n}$.

Let $g_x = \max\{h_{y_1}, h_{y_2}, \ldots \ldots h_{y_n}\}$ then $g_x \in B$ (by step 2)

and $g_x(x) = f(x)$ $[\because h_{y_i}(x) = f(x) \ \forall i]$

and if $t \in K$ then $\exists$ some i such that $t \in J_{y_i}$

$-\epsilon < (h_{y_i} - f)(t) < \epsilon$

$\Rightarrow h_{y_i}(t) > f(t) - \epsilon$

$\Rightarrow g_x(t) > h_{y_i}(t) > f(t) - \epsilon$.

Step 4: Given any continuous fxn on compact set K and $\epsilon > 0$, $\exists$ $h \in B$ st $|h(x) - f(x)| < \epsilon \ \forall x \in K$.

Consider the functions g_x for each $x \in K$ (constructed in step 3)

Given $\epsilon > 0$ $\exists$ an open set $V_x \ni x$ st

$-\epsilon < (g_x - f)(t) < \epsilon \ \forall t \in V_x$

In particular,

$g_x(t) < f(t) + \epsilon \ \forall t \in V_x$

Now, $K \subseteq \bigcup_{x \in K} V_x$

as K is compact, ∃ $x_1, x_2, \ldots x_m \in K$ s.t

$K \subseteq V_{x_1} \cup \cdots \cup V_{x_m}$

~~Define~~ Let $h = \min \{g_{x_1}, \ldots g_{x_m}\}$
then $h \in B$
& $h(t) > f(t) - \epsilon \quad \forall t \in K$

Now if $t \in K \Rightarrow t \in V_{x_j}$ for some $j \in \{1, \ldots m\}$

$\Rightarrow h(t) < g_{x_j}(t) < f(t) + \epsilon$

$\Rightarrow h(t) - f(t) < \epsilon \quad \forall t \in K$

$\Rightarrow |h(t) - f(t)| < \epsilon \quad \forall t \in K$

$\Rightarrow f(t) \in \bar{B} = B$

$\Rightarrow f \in B$

Hence proved

Functions of Several Variables

Def^n If X and Y are finite vector spaces over $\mathbb{R}$ and if $A: X \to Y$ is said to be a linear transformation if

$$A(\alpha x_1 + \beta x_2) = \alpha A(x_1) + \beta A(x_2) \quad \forall x_1, x_2 \in X, \alpha, \beta \in \mathbb{R}$$

Def^n If $A: X \to X$ is a linear transformation, then A is called linear operator.

$L(X, Y)$ = set of all LT from X to Y.
$L(X)$ = set of all LO from X to X.

Theorem → If $A \in L(X)$, where $\dim X < \infty$ then A is 1-1 iff A is onto.

Def^n (1) If $A_1, A_2 \in L(X, Y)$ & $c_1, c_2 \in \mathbb{R}$

define $c_1A_1 + c_2A_2$ by
$$(c_1A_1 + c_2A_2)(x) = c_1A_1(x) + c_2A_2(x)$$

To prove → $c_1A_1 + c_2A_2 \in L(X, Y)$
i.e., $(c_1A_1 + c_2A_2)(\alpha x_1 + \beta x_2) = \alpha(c_1A_1 + c_2A_2)(x_1) + \beta(c_1A_1 + c_2A_2)(x_2)$

Def^n (2) If X, Y, Z are vector spaces and $A \in L(X, Y)$, $B \in L(Y, Z)$ then we define their product BA as $(BA)(x) = B(A(x))$ then

Functions of several variables

Defn If X and Y are finite vector spaces over $\mathbb{R}$ and if $A : X \to Y$ is said to be a linear transformation if

$$A(\alpha x_1 + \beta x_2) = \alpha A(x_1) + \beta A(x_2)$$
$$\forall\ x_1, x_2 \in X,\ \alpha, \beta \in \mathbb{R}$$

Defn If $A : X \to X$ is a linear transformation, then A is called linear operator.

$L(X, Y)$ = set of all LT from X to Y.
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IMP
Theorem → If $A \in L(X)$, where $\dim X < \infty$ then A is 1-1 iff A is onto.

IMP
Defn ① If $A_1, A_2 \in L(X, Y)$ & $c_1, c_2 \in \mathbb{R}$

define $c_1 A_1 + c_2 A_2$ by
$$(c_1 A_1 + c_2 A_2)(x) = c_1 A_1(x) + c_2 A_2(x)$$

To prove → $c_1 A_1 + c_2 A_2 \in L(X, Y)$
i.e., $(c_1 A_1 + c_2 A_2)(\alpha x_1 + \beta x_2) =$
$\alpha (c_1 A_1 + c_2 A_2)(x_1) +$
$\beta (c_1 A_1 + c_2 A_2)(x_2)$

(2) If X, Y, Z are vector spaces and $A \in L(X, Y)$, $B \in L(Y, Z)$ then we define their product BA as $(BA)(x) = B(A(x))$ then

$BA \in L(X, Z)$

To prove: $BA(\alpha x_1 + \beta x_2) = \alpha(BA)(x_1) + \beta(BA)(x_2)$

$\forall\, x_1, x_2 \in X;\ c_1, c_2 \in R$

(3) For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$. Define the norm $\|A\|$ of A by

$$\|A\| = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\| \leq 1}} \|Ax\|$$

Notice that $\|Ax\| \leq \|A\| \|x\|$

$\left[\because \|A\| = \sup_{\|x\| \leq 1} \|Ax\| \right.$

$\Rightarrow \|A\| \geq \dfrac{\|Ax\|}{\|x\|}$

$\Rightarrow \|A\| \geq \dfrac{1}{\|x\|} \|Ax\|$

$\left. \|A\| \|x\| \geq \|Ax\| \right]$

Result: (i) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and c is a scalar, then

$\|A + B\| \leq \|A\| + \|B\|$ and

$\|cA\| = |c| \|A\|$

(ii) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$ then

$\|BA\| \leq \|B\| \|A\|$

Proof: Define a metric d on $L(\mathbb{R})$ by

$d : L(X) * L(X) \longrightarrow \mathbb{R}$

$d(A, B) = \|A - B\|$

Let $A - B = T$

$d(A, B) = \|T\| = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\| \leq 1}} \geq 0$

and if $A = B \Rightarrow T = 0$

$\Rightarrow T(x) = 0 \ \forall\, x \in \mathbb{R}^n$

$\Rightarrow \|T\| = 0$

$\Rightarrow d(A, B) = 0$

If $d(A, B) = 0 \Rightarrow \|T\| = 0$

$\Rightarrow \sup_{\substack{x \in \mathbb{R}^n \\ \|x\| \leq 1}} \|Tx\| = 0$

$\Rightarrow \|Tx\| = 0 \ \forall\, x \in \mathbb{R}^n, \|x\| = 1$

If $y \in \mathbb{R}^n$ st $y \neq 0$, then $\left\| T \dfrac{y}{\|y\|} \right\| = 0$

$\Rightarrow \dfrac{\|Ty\|}{\|y\|} = 0$

$\Rightarrow \|Ty\| = 0$

$\Rightarrow Ty = 0$

$\Rightarrow T(y) = 0 \ \forall \ y \in \mathbb{R}^n \setminus \{0\}$

also, $T(0) = 0$

$\Rightarrow Ty = 0 \ \forall \ y \in \mathbb{R}^n$

$\Rightarrow T = 0$

$\Rightarrow \boxed{A = B}$

(ii) $d(A, B) = \|T\| = \sup_{x \in \mathbb{R}^n} \|Tx\|$

$d(B, A) = \|-T\| = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\| \le 1}} \|-T(x)\|$

$= \cancel{\sup_{\substack{x \in \mathbb{R}^n \\ \|x\| \le 1}} \|T(-x)\|}$

$= \|T\|$

$= d(A, B)$

(iii) $d(A, C) + d(C, B) \geq d(A, B)$

i.e., $\underbrace{\|A - C\|}_{T_1} + \underbrace{\|C - B\|}_{T_2} \geq \|A - B\|$

Derivative of Several Variables →

Def^n Let $E \subseteq \mathbb{R}^n$ be open and $f : E \to \mathbb{R}^m$ we say f is differentiable at $x \in E$ if $\exists A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that $\Rightarrow f(x+h) - f(x) = Ah + r(h)$, $h \in \mathbb{R}^n$ where $\lim_{h \to 0} \frac{\|r(h)\|}{\|h\|} = 0$ and we write $A = f'(x)$.

here $h \in \mathbb{R}^n$, $r(h) \in \mathbb{R}^m$

Mean Value Theorem :-

Suppose $f:[a,b] \to \mathbb{R}^m$ is continuous and differentiable on (a,b). Then $\exists$ $x \in (a,b)$ such that

$$\|f(b)-f(a)\| \le (b-a)\|f'(x)\|$$

Proof: Let $z = f(b)-f(a)$

If $z=0$, nothing to prove.

So let $z \neq 0$, Define $\phi:[a,b] \to \mathbb{R}$ st

$$\phi(t) = z\cdot f(t) \quad [\text{inner product in } \mathbb{R}^m]$$

ϕ is continuous on $[a,b]$ and diff on (a,b)

($\because f$ is continuous on $[a,b]$ & diff on (a,b))

So by LMV on ϕ, we get

$\exists$ $x \in (a,b)$ St

$$\phi'(x) = \frac{\phi(b)-\phi(a)}{b-a}$$

$$z\cdot f'(t)(b-a) = \phi(b)-\phi(a) \quad \text{—— (1)}$$

Now, $\phi(b)-\phi(a) = z\,f(b) - z\cdot f(a)$
$= z(f(b)-f(a))$
$= z\cdot z = \|z\|^2$

Put in equation (1), we get

$$\|z\|^2 = z\,f'(x)(b-a) \quad \text{—— (2)}$$

$\|zf'(x)\| \le \|z\|\,\|f'(x)\|$ (So from eq^n 2)

$$\|z\|^2 \le \|z\|\,\|f'(x)\|\,(b-a)$$

$$\|z\| \le \|f'(x)\|\,(b-a)$$

$$\|f(b)-f(a)\| \le \|f'(x)\|(b-a)$$

Hence proved

Convex functions :-

Def^n: Let $f:[a,b] \to \mathbb{R}$ be a function, then f is called convex if $\forall$ $0 \le t \le 1$,

~~$f(ta+(1-t)b) \le t f(a) + (1-t) f(b)$~~

$$f((1-t)a+tb) \le (1-t)f(a) + t f(b)$$

Theorem :- Suppose f maps a convex open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E, and $\exists$ a real number M such that $\|f'(x)\| \le M$ $\forall$ $x \in E$. then,

$$\|f(b)-f(a)\| \le M\|b-a\| \quad \forall\ a,b \in E.$$

Note:- open interval means no element include while hold.

Proof: fix $a, b \in E$

Define $\gamma(t) = ta + (1-t)b \;\forall\; t \in \mathbb{R}$ for which $\gamma(t) \in E$.

as E is convex $\Rightarrow \gamma(t) \in E$ if $0 \le t \le 1$

Put $g(t) = f(\gamma(t))$ i.e., $g : [0,1] \to \mathbb{R}^m$

as f and γ are differentiable, g is also differentiable.

$g'(t) = f'(\gamma(t))\,\gamma'(t) = f'(\gamma(t))\,(a-b)$

$\Rightarrow \|g'(t)\| = \|f'(\gamma(t))\|\;\|a-b\|$
$\le M\,\|a-b\|$

Now apply previous theorem on $g(t)$,

$\|g(1) - g(0)\| \le \|g'(t)\|\,(1-0)$

$\|f(a) - f(b)\| \le M\,\|a-b\| \quad \forall\, a, b \in E$

Definition:- Let X be a metric space with metric d. Let

$\phi : X \to X$ satisfies $d(\phi(x), \phi(y)) \le \alpha\, d(x,y) \;\forall\, x, y \in X$ and $\alpha \in [0,1)$ then ϕ is said to be a contradiction.

→ Contraction mapping

Remark:- ϕ is uniformly continuous.

Contraction Mapping Theorem:-

Let X be a complete metric space. If $\phi : X \to X$ is a contraction, then $\exists$ one and only one $x \in X$ such that $\phi(x) = x$.

Proof: As ϕ is contradiction

$\Rightarrow d(\phi(x), \phi(y)) \le \alpha\, d(x,y) \;\forall\, x, y \in X$ where $\alpha \in [0,1)$.

Case 1: If $\alpha = 0 \Rightarrow d(\phi(x), \phi(y)) \le 0 \;\forall\, x, y \in X$

$\Rightarrow d(\phi(x), \phi(y)) = 0 \;\forall\, x, y \in X$

$\Rightarrow \phi(x) = \phi(y) \;\forall\, x, y \in X$.

$\Rightarrow \phi$ is a constant function say $\phi(x) = c \;\forall\, x \in X$ then

$\phi(c) = c$ and c is the only fixed element.

So, now let $\alpha \ne 0$.

Let $x_0 \in X$ be arbitrary

$\phi(x_0) = x_1$ (say), $\phi(x_1) = x_2$ (say) and so on

$\phi(x_n) = x_{n+1}$.

$$d(x_{n+1}, x_n) = d(\phi(x_n), \phi(x_{n-1}))$$
$$\le \alpha d(x_n, x_{n-1})$$
$$= \alpha d(\phi(x_{n-1}), \phi(x_{n-2}))$$
$$\le \alpha^2 d(x_{n-1}, x_{n-2}) \text{ and so on,}$$

we get
$$\boxed{d(x_{n+1}, x_n) \le \alpha^n d(x_1, x_0)}$$

for $n < m$

$$d(x_n, x_m) = d(x_m, x_n) \le d(x_m, x_{m-1}) + d(x_{m-1}, x_n)$$
$$\le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + d(x_{m-2}, x_n)$$
$$\le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)$$
$$\le \alpha^{m-1} d(x_1, x_0) + \alpha^{m-2} d(x_1, x_0) + \cdots \alpha^n d(x_1, x_0)$$
$$= (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n)\, d(x_1, x_0)$$

GP
$$= (\alpha^n + \alpha^{n+1} + \cdots \alpha^{m-2} + \alpha^{m-1})\, d(x_1, x_0)$$

$\frac{a(1-x^n)}{1-x}$

$$= \frac{\alpha^n(-1 + (\frac{1}{\alpha})^{m-n})}{-1 + \frac{1}{\alpha}}\, d(x_1, x_0)$$

$$d(x_n, x_m) = \frac{\alpha^n(1 - (\alpha)^{m-n})}{\alpha^{m-n}\frac{(1-\alpha)}{\alpha}}\, d(x_1, x_0)$$

$$d(x_n, x_m) = \frac{\alpha^n(1 - \alpha^{m-n})}{1-\alpha}\, d(x_1, x_0)$$
$$\longrightarrow 0 \text{ as } n \to \infty$$

$x_0, x_1, x_2 \ldots$ is Cauchy sequence

as X is complete $\Rightarrow \{x_n\} \longrightarrow x \in X$

as ϕ uniformly continuous

$$\Rightarrow \{\phi(x_n)\} \longrightarrow \phi(x)$$
$$\Rightarrow \{x_{n+1}\} \longrightarrow \phi(x)$$

but $\{x_n\} \longrightarrow x \Rightarrow \phi(x) = x$.

<u>Uniqueness</u>:- Let $\phi(x) = x$ and $\phi(y) = y$ for some $x, y \in X$.

$$d(\phi(x), \phi(y)) = d(x, y) \le \alpha d(x, y)$$
$$\Rightarrow\Leftarrow \text{ as } \alpha \in [0,1).$$

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<u>Inverse function Theorem</u>:- Suppose f is a C' mapping of an open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$, $f'(a)$ is invertible for some $a \in E$, $b = f(a)$. Then

(i) $\exists$ open sets U and V in $\mathbb{R}^n$ st $a \in U$, $b \in V$, f is one-one on U & $f(U) = V$

(ii) If g is the inverse of f defined in V by $g(f(x)) = x$, $x \in U$, then $g \in C'(V)$

Proof, i.e., in the nbd of a we find the inverse i.e, local inverse may exist but Global inverse may not exist. Here, $g(x)$ is the local inverse.

Proof: Let $f'(a) = A$ and choose $\lambda > 0$ such that

$$2\lambda \|A^{-1}\| = 1$$

i.e, $\lambda = \frac{1}{2\|A^{-1}\|}$ (A^{-1} exists as $f'(a)$ is invertible)

($\because A = f'(a)$) Since, f' is continuous at a, for $\lambda > 0$ $\exists$ $\delta > 0$ such that $\|f'(x) - A\| < \lambda$ for $x \in U = B(a, \delta)$ —— (1)

For each $y \in \mathbb{R}^n$ defined ϕ by

$$\phi(x) = x + A^{-1}(y - f(x)), x \in E \quad \text{— (A)}$$

This will help to show that f is one-one on U i.e. for each $x \in U$ $\Rightarrow$ unique $f(x)$ inv.

Note that $f(x) = y$ iff $\phi(x) = x$ —— (B)

or $y = f(x)$ iff x is a fixed pt of ϕ.

Now, for $h \in \mathbb{R}^n$

$$\phi(x+h) - \phi(x) = x + h - A^{-1}(y - f(x+h)) - x - A^{-1}(y - f(x))$$
$$= h - A^{-1}(f(x+h) - f(x))$$
$$= h - A^{-1}(f'(x)h + r(h))$$
$$= A^{-1}Ah - A^{-1}(f'(x))h$$

where $\lim_{h \to 0} \frac{\|r(h)\|}{\|h\|} = 0$ as f is diff

$\therefore \phi(x+h) - \phi(x) = A^{-1}(Ah - f'(x)h) - A^{-1}r(h)$

$$= (I - A^{-1}f'(x))h - A^{-1}r(h)$$

where $\lim_{h \to 0} \frac{\|A^{-1}r(h)\|}{\|h\|} = 0$

as $\|A^{-1}\|$ is a fixed no.

$\Rightarrow \phi$ is differentiable and $\phi'(x) = I - A^{-1}f'(x)$

$$\|\phi'(x)\| = \|I - A^{-1}f'(x)\|$$
$$= \|A^{-1}A - A^{-1}f'(x)\|$$
$$\le \|A^{-1}\| \|A - f'(x)\|$$
$$\le \lambda \|A^{-1}\| = \frac{1}{2} < 1, x \in U$$

$\therefore \|\phi(x_1) - \phi(x_2)\| \le \|\phi'(x)\| \|x_1 - x_2\|$ for $x_1, x_2 \in U$

By Mean Value theorem for vector valued function

$\Rightarrow \|\phi(x_1) - \phi(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|$ —— (*)

$\Rightarrow \phi$ has atmost one fixed pt.

If not then $\phi(x_1) = x_1$, $\phi(x_2) = x_2$ and from (*)

$\|x_1 - x_2\| \le \frac{1}{2}\|x_1 - x_2\|$ which is absurd

$\therefore y = f(x)$ for atmost one $x \in U$ (from A and B)

$\therefore f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$ in U

$\Rightarrow f$ is one-one on U.

Put $V = f(U)$,

We show V is open

Let $y_0 \in V$

then $y_0 = f(x_0)$ for some $x_0 \in U$.

Let $B = B(x_0, r)$ be such that $r > 0$ is so small so that $\bar{B} \subset U$.

Claim: $B(y_0, \lambda r) \subset V$

Let $y \in B(y_0, \lambda r)$ be fixed

Then with this y in the definition of ϕ

$$\|\phi(x_0) - x_0\| = \|x_0 + A^{-1}(y - f(x_0)) - x_0\|$$

$$\leq \|A^{-1}\| \, \|y - y_0\| \quad [\because f(x_0) = y_0]$$

$$\leq \frac{1}{2\lambda} \lambda r = \frac{r}{2} \quad \text{— (C)}$$

We show $\phi : \bar{B} \to \bar{B}$ If $x \in \bar{B}$

Then $\|\phi(x) - x_0\| \leq \|\phi(x) - \phi(x_0)\| + \|\phi(x_0) - x_0\|$

$$\leq \frac{1}{2} \|x - x_0\| + \frac{r}{2}$$

$$\leq \frac{r}{2} + \frac{r}{2} \quad \left[\text{from (C) and as } \|x - x_0\| \leq r,\ x_0, x \in \bar{B} \subset U \right]$$

$\Rightarrow \phi(x) \in B(x_0, r)$

$\Rightarrow \phi(x) \in \bar{B}$

Thus ϕ is a contradiction from $\bar{B}$ to $\bar{B}$ which is complete ($\because$ a closed set is always complete)

So, ϕ has a unique fixed pt in $\in \bar{B}$

(By Contraction mapping theorem)

So, $x \in \bar{B}$

i.e., $\exists\ x \in \bar{B}$ st $\phi(x) = x$

$\Rightarrow y = f(x)$ for $x \in \bar{B} \subset U$

$\Rightarrow y \in f(\bar{B}) \subset f(U) = V$

(as f is continuous so $\bar{B} \in U$

$\Rightarrow f(\bar{B}) \subset f(U)$

$\Rightarrow B(y_0, \lambda r) \in V \Rightarrow V$ is open.

(b) Let $y, y+k \in V$

$\Rightarrow x, x+h \in U$ st $f(x) = y$ &

$f(x+h) = y+k$

Then for this y in definition of ϕ

$$\phi(x+h) - \phi(x) = x+h+A^{-1}[y - f(x+h)] - x - A^{-1}[y - f(x)]$$

$$= h - A^{-1}[f(x+h) - f(x)]$$

$$= h - A^{-1}k$$

Since, $x+h, x \in U$, we have

$$\|h - A^{-1}k\| = \|\phi(x+h) - \phi(x)\|$$

$$\leq \frac{1}{2}\|x+h-x\|$$

$$= \frac{1}{2}\|h\|$$

$\Rightarrow \|A^{-1}k\| = \|h + A^{-1}k - h\| \geq \|h\| - \|A^{-1}k - h\|$

Apply theorem of If Ω be the set of all invertible operators on R^n & if $A \in \Omega$, $B \in L(R^n)$ & $\|B - A\| < \frac{1}{\|A^{-1}\|}$ then $B \in \Omega$

$$\geq \|h\| - \frac{1}{2}\|h\| = \frac{1}{2}\|h\|$$

$$\Rightarrow \|A^{-1}k\| \geq \frac{1}{2}\|h\|$$

$$\therefore \|h\| \leq 2\|A^{-1}k\|$$

$$\leq 2\|A^{-1}\|\,\|k\|$$

$$= \frac{1}{\lambda}\|k\| \quad \left(\because \lambda = \frac{1}{2\|A^{-1}\|}\right)$$

$$\Rightarrow \|h\|\lambda \leq \|k\|$$

$$\Rightarrow \frac{1}{\|k\|} \leq \frac{1}{\|h\|\lambda}$$

This implies that $h \to 0$ whenever $k \to 0$

Since, $\|f'(x) - A\| < \frac{1}{2\|A^{-1}\|} < \frac{1}{\|A^{-1}\|}$ (By eq^n 1)

and A is invertible, we get $f'(x)$ is invertible (By theorem (*))

Let $T = (f'(x))^{-1}$

Now, $g(y+k) - g(y) - Tk =$

$$g(f(x+h)) - g(f(x)) - Tk$$

$$= x+h-x-Tk \quad [\text{By def}^n \text{ of } g]$$

$$= h - Tk$$

$$= h - T(f(x+h) - f(x))$$

$$(\because k = f(x+h) - y)$$

$$= -T[f(x+h) - f(x) - T^{-1}h]$$

$$= -T[f(x+h) - f(x) - f'(x)h]$$

$$[\because T = (f'(x))^{-1} \Rightarrow T^{-1} = f'(x)]$$

$$\Rightarrow \frac{\|g(y+k) - g(y) - Tk\|}{\|k\|} \leq \frac{\|T\| \, \|f(x+h) - f(x) - f'(x)h\|}{\|k\|}$$

$$\leq \frac{\|T\|}{\lambda} \frac{\|f(x+h) - f(x) - f'(x)h\|}{\|h\|}$$

The RHS $\to 0$ as $k \to 0$, so $h \to 0$

$\therefore$ g is differentiable

$\therefore$ g is differentiable at y and

$$g'(y) = T$$

$$= (f'(x))^{-1}$$

$$= (f'(g(y)))^{-1}, \; y \in V.$$

Since g is continuous (being differentiable) & f' is continuous as f' is continuous from E to $L(\mathbb{R}^n)$ and $A \to A^{-1}$ is continuous on the set of all invertible operators in $L(\mathbb{R}^n)$ we get $g \in C'(V)$ (as f' and g are continuous so $(f'g(y))^{-1}$ is continuous]

Implicit function Theorem :-

Let f be a C'-mapping of an open set $E \subseteq \mathbb{R}^{n+m}$ into $\mathbb{R}^n$ such that $f(a,b) = 0$ for some $(a,b) \in E$. Put $A = f'(a,b)$ and assume that A_x is invertible then $\exists$ open set $U \subseteq \mathbb{R}^{n+m}$, $W \subseteq \mathbb{R}^m$ with $(a,b) \in V$ and $b \in W$ having the following property. To every $y \in W$, there corresponds a unique x such that $(x,y) \in U$ and $f(x,y) = 0$

If x is defined as $g(y)$ then g is a C' mapping of W into $\mathbb{R}^n$, $g(b) = a$

and $f(g(y), y) = 0$ for $y \in W$

$$g'(b) = (-A_x)^{-1} A_y$$

only statement

Define $F : E \to \mathbb{R}^{n+m}$ by

$$F(x,y) = (f(x,y), y)$$

Here, $f : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$

first n components of y remaining are

$y = y_1 \text{———} y_m$.

The matrix $F'(x,y)$ is $\longrightarrow$

$$\begin{vmatrix} D_1 f_1 & D_n f_1 & D_{n+1} f_1 & D_m f_1 \\ D_1 f_2 & D_n f_2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ D_1 f_n & D_n f_n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{pmatrix} (D_j f_i)_{n\times n} & O_{n\times m} \\ O_{m\times n} & I_{m\times n} \end{pmatrix} \begin{bmatrix} A_x & A_y \\ O_{m\times n} & I_{m\times n} \end{bmatrix}$$

Since, $f \in e'$ and zeroes are also continuous, we get $F \in e'$.

Claim: $F'(a,b)$ is invertible in $L(\mathbb{R}^{n+m})$

$f(a+h, b+k) - f(a,b) = f'(a,b)(h,k) + r(h,k)$ where $(h,k) \in \mathbb{R}^{n+m}$

& $\lim_{\|h,k\| \to 0} \frac{\|r(h,k)\|}{\|h,k\|} \to 0$

Since $f(a,b) = 0$

$\Rightarrow f(a+h, b+k) = A(h,k) + r(h,k)$ $[\because A = f'(a,b)]$

Now, $F(a+h, b+k) - F(a,b)$

$= (f(a+h, b+k), b+k) - (f(a,b), b)$

$= (f(a+h, b+k), k)$ $[\because f(a,b) = 0]$

$= (A(h,k) + r(h,k), k)$

$= (A(h,k), k) + (r(h,k), 0)$

Since, $\lim_{\|h,k\| \to 0} \frac{\|r(h,k)\|}{\|h,k\|}$

$= \lim_{\|h,k\| \to 0} \frac{\|r(h,k)\|}{\|h,k\|} = 0$

we get,

$F'(a,b)(h,k) = (A(h,k), k)$

Therefore, $F'(a,b)$ is one-one as

$F'(a,b)(h,k) = 0$

$\Rightarrow A(h,k) = 0, k = 0$

(Here 0 is a vector)

$A(h,0) = 0$

$\Rightarrow A_x h + A_y 0 = 0$

$\Rightarrow h = 0$ as A_x is invertible.

Thus $F'(a,b)$ is one-one and so onto (By Thm)

Thus our claim is proved.

i.e. $F'(a,b)$ is invertible.

Now, applying inverse function theorem to F at (a,b)

$\exists$ open set $U \subseteq E$, V in $\mathbb{R}^{n+m}$ st $(a,b) \in U$, $(0,b) \in V$, $F(U) = V$ & F is one-one and $G: F^{-1}: V \longrightarrow U$ in e'.

Let $W = \{y \in \mathbb{R}^m : (0, y) \in V\}$

then $W \neq \phi$ as $b \in W$, Also W is open as W is the inverse image of V under the continuous map $\mathbb{R}^n \to \mathbb{R}^m$: $y \to (0, y)$

Proof of Property: Let $y \in W$, then

$(0, y) \in V = F(U)$

$\Rightarrow \exists\, (x, y) \in U$ st $(0, y) = F(x, y)$
$= (f(x, y), y)$

i.e., $f(x, y) = 0$

To show uniqueness of x, let $(x', y) \in U$ be st $f(x', y) = 0$

Then $(0, y) = \begin{cases} (f(x, y), y) = F(x, y) \\ f(x', y), y) = F(x', y) \end{cases}$

Since F is one-one on U, $x = x'$

For $y \in W$, write $g(y)$ for the unique x st $(x, y) \in U$ & $f(x, y) = 0$ i.e.,
$f(g(y), y) = 0$, $y \in W$.

Therefore,
$F(g(y), y) = (0, y)$, $y \in W$
[as $f(g(y), y) = 0$]
$\Rightarrow (g(y), y) = G(0, y)$ [as $(0, y) \in V$, $F^{-1}: G: U \to U$]

Since, $G \in C'$ $(D_{n+j}\, g_i)$ $1 \leq i \leq n$, $1 \leq j \leq m$ are continuous

(Here $g = (g_1, g_2, \cdots g_n)$
$D_{n+j} = \frac{\partial}{\partial y_j}$)

We get $g \in C'$

Also, for $y = b$, $(0, b) = F(g(b), b)$
$= (f(g(b), b), b)$
$= (f(a, b), b)$ [$\because f(a, b) = 0$]

By uniqueness of x, we get $g(b) = a$

Finally, to compute $g'(b)$ put

$\phi(y) = (g(y), y)$, $y \in W$

then $\phi \in C'(W)$ and

$\phi(y+k) - \phi(y) = (g(y+k), y+k) - (g(y), y)$
$= (g(y+k) - g(y), k)$
$= (g'(y) \cdot k + r_g(k), k)$
$= (g'(y)k, k) + (r_g(k), 0)$

$\therefore \phi'(y)k = (g(y)k, k)$ as $\lim_{k \to 0} \frac{\|(r_g(k), 0)\|}{\|k\|} = 0$

Now, $0 = f(g(y), y)$, $y \in W, k \in \mathbb{R}^m$
$= f(\phi(y))$

By chain rule for differentiation →

$f'(\phi(y))\,\phi'(y) = 0$

$\Rightarrow f'(\phi(y))\,\phi'(y)\cdot k = 0$

For $y = b$, $\phi(b) = (a, b)$ (as $g(b) = a$)

$f'(a,b)\,(g'(b)k, k) = 0 \quad \forall\, k \in \mathbb{R}^m$

or $A(g'(b)k, k) = 0 \;\forall\; k \in \mathbb{R}^m$ [$\because A = f'(a,b)$]

or $A_x g'(b)k + A_y k = 0$

or $A_x g'(b) + A_y = 0$

$$\boxed{\Rightarrow g'(b) = -(A_x)^{-1} A_y}$$

Assignment ② a) Let $x \in f^{-1}(\cup Y_\alpha)$

$\Rightarrow f(x) \in \cup Y_\alpha$
$\Rightarrow f(x) \in Y_\alpha$ for some $\alpha \in \Lambda$
$\Rightarrow x \in f^{-1}(Y_\alpha)$ for some $\alpha \in \Lambda$

$\Rightarrow x \in \bigcup_{\alpha\in\Lambda} f^{-1}(Y_\alpha)$

Now, let $y \in \bigcup_{\alpha\in\Lambda} f^{-1}(Y_\alpha)$

$\Rightarrow y \in f^{-1}(Y_\alpha)$ for some $\alpha \in \Lambda$
$\Rightarrow f(y) \in Y_\alpha$

$\Rightarrow f(y) \in \bigcup_{\alpha\in\Lambda} Y_\alpha$

$y \in f^{-1}(\bigcup_{\alpha\in\Lambda} Y_\alpha)$

$f^{-1}(\cup Y_\alpha) = \bigcup_{\alpha\in\Lambda} f^{-1}(Y_\alpha)$

b) Let $x \in f^{-1}(\cap Y_\alpha)$
$\Rightarrow f(x) \in \cap Y_\alpha$
$\Rightarrow f(x) \in Y_\alpha$ for all $\alpha \in \Lambda$
$\Rightarrow x \in f^{-1}(Y_\alpha)$ for all $\alpha \in \Lambda$

$\Rightarrow x \in \bigcap_{\alpha\in\Lambda} f^{-1}(Y_\alpha)$

Conversely → Now, let $y \in \bigcap_{\alpha\in\Lambda} f^{-1}(Y_\alpha)$

$y \in f^{-1}(Y_\alpha)$ for all $\alpha \in \Lambda$

$f(y) \in Y_\alpha$

$\Rightarrow f(y) \in \bigcap_{\alpha\in\Lambda} Y_\alpha$

$y \in f^{-1}(\cap Y_\alpha)$

$f^{-1}(\cap Y_\alpha) = \bigcap_{\alpha\in\Lambda} f^{-1}(Y_\alpha)$

c) Let $y \in f(\bigcup_{\alpha\in\Lambda} X_\alpha)$

$\Rightarrow \exists$ some $x \in \bigcup_{\alpha\in\Lambda} X_\alpha$ st $y = f(x)$

$\Rightarrow x \in X_\alpha$ for some $\alpha \in \Lambda$ st

$y = f(x)$

$\Rightarrow y \in f(X_\alpha)$ for some $\alpha \in \Lambda$ st,

$\Rightarrow y \in \bigcup_{\alpha \in \Lambda} f(X_\alpha)$

Conversely Now, Let $z \in \bigcup_{\alpha \in \Lambda} f(X_\alpha)$

$\Rightarrow z \in f(X_\alpha)$ for some $\alpha \in \Lambda$

$\Rightarrow \exists$ some $x \in X_\alpha$ st

$z = f(x)$

$\Rightarrow \exists$ some $x \in \bigcup_{\alpha \in \Lambda} X_\alpha$ st,

$z = f(x)$

$\Rightarrow z \in f\left(\bigcup_{\alpha \in \Lambda} X_\alpha\right)$

d) Let $y \in f(\cap X_\alpha)$

$\Rightarrow \exists$ some $x \in \bigcap_{\alpha \in \Lambda} X_\alpha$ st

$y = f(x)$

$\Rightarrow x \in X_\alpha$ for all $\alpha \in \Lambda$

$y = f(x)$

$\Rightarrow y \in f(X_\alpha)$ for all $\alpha \in \Lambda$

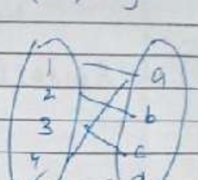
$\Rightarrow y \in \bigcap_{\alpha \in \Lambda} f(X_\alpha)$

Example:) $X = \{1, 2, 3, 4, 5\}$

$X_1 = \{1, 2, 3\}$ $\quad X_2 = \{3, 4, 5\}$

$f(X_1 \cap X_2) = f(3)$

$= c$



RHS $f(X_1) \cap f(X_2) =$

$\{a, b, c\} \cap \{c, a, d\}$

$= \{c, a\}$

2nd Part (4) To prove $\rightarrow$ Equality holds $\Rightarrow f$ is 1-1

or f is not 1-1 $\Rightarrow$ Equality does not hold.

Assume that equality holds,

TP f is 1-1

Suppose f is not 1-1

$\Rightarrow \exists\, x_1, x_2 \in X$ st $x_1 \neq x_2$ but

$f(x_1) = f(x_2) = y$

Take $X_1 = \{x_1\}$ $\quad X_2 = \{x_2\}$

then $f(x_1) = y$, $f(x_2) = y$

RHS $\Rightarrow f(x_1) \cap f(x_2) = \{y\}$

and $f(x_1 \cap x_2) = f\{\phi\} = \phi$

$\Rightarrow$ RHS $\neq$ LHS

which is a contraction

$\therefore$ Equality holds iff it is one-one