2 Infinite Series and Series of Arbitrary Terms

Art. 1. Definitions

Infinite Series. Let $\{a_n\}$ be any given sequence. Then an expression of the form

$$a_1 + a_2 + a_3 + \dots\dots + a_n + \dots\dots$$

is called an Infinite series and is denoted by $\sum_{n=1}^{\infty} a_n$ or simply by $\sum a_n$.

Finite Series. Let m be a positive integer, then

$$\sum_{n=1}^{m} a_n = a_1 + a_2 + a_3 + \dots\dots\dots + a_m$$

is called a finite series.

Note. We shall deal with Infinite series only. Hence by series we mean Infinite series.

Terms of series. $a_1, a_2, \dots\dots, a_n, \dots\dots$ are called the first, sceond,........, nth,........terms of the infinite series.

General Term of series. a_n is called the general term of the series.

Partial Sums. Let $\Sigma a_n = a_1 + a_2 + \dots\dots + a_n + \dots\dots$ be the given series.

Let $S_1 = a_1$, $\quad S_2 = a_1 + a_2$, $\quad S_3 = a_1 + a_2 + a_3$,

... ... $\quad$ ... ... $\quad$ ... ...

$S_n = a_1 + a_2 + \dots\dots\dots + a_n$ and so on.

Then $S_1, S_2, S_3, \dots\dots\dots, S_n, \dots\dots\dots$ are called the first, second, third,.........., nth,............partial sums of the given series.

Sequence of Partial sums. Let $S_n = a_1 + a_2 + \dots\dots + a_n$, $\forall\, n \in N$

Then the sequence $\{S_n\}$ is called the **Sequence of Partial Sums** of the series Σa_n.

Dependence of Series on Sequence of Partial Sums.

The series Σa_n and sequence of partical sums $\{S_n\}$ behave alike *i.e.* Σa_n and $\{S_n\}$ have the same behaviour.

Behaviour and sum of series. The series Σa_n is said to converge, diverge or oscillate according as the sequence $\{S_n\}$ of partial sums converges, diverges or oscillates.

Let the given series be

$$\Sigma a_n = a_1 + a_2 + \dots\dots\dots + a_n + \dots\dots\dots$$

and $\quad S_n = a_1 + a_2 + \dots\dots\dots + a_n$

and suppose $\quad \underset{n\to\infty}{\text{Lt}} S_n = S$

i.e., $\{S_n\}$ converges to S.

Then S is called the sum of the series Σa_n and we write $\Sigma a_n = S$

Convergent series. The series Σa_n is said to be **convergent** if $\underset{n\to\infty}{\text{Lt}} S_n = S$ (finite) and S is called the sum of the convergent infinite series Σa_n.

Divergent series. The series Σa_n is said to be **divergent** to ∞ or $-\infty$ if $\underset{n\to\infty}{\text{Lt}} S_n = \infty$ or $-\infty$ and we write $\Sigma a_n = \infty$ or $-\infty$.

Note. The sum of a divergent series does not exist.

Oscillatory series. The series Σa_n is said to be **oscillatory** if $\{S_n\}$ is oscillatory *i.e.* if the sequence $\{S_n\}$ is neither convergent to *a* finite limit nor divergent to ∞ or $-\infty$.

The series Σa_n **oscillates finitely or infinitely** according as the sequence is bounded or unbounded.

Non-convergent series. The series which diverges or coscillates is called a **convergent series.**

Example 1. *Discuss the convergence of the series*

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots\ldots$$

Sol. Here $\quad S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots\ldots+\frac{1}{2^{n-1}}$

$$= \frac{1\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = 2\left[1-\frac{1}{2^n}\right]$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ S_n = \underset{n\to\infty}{\text{Lt}}\ 2\left[1-\frac{1}{2^n}\right] = 2(1-0) = 2$$

Hence the sequence $\{S_n\}$ converges to 2

Therefore, the given series converges and its sum is 2.

Example 2. *Examine the series*

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 + \ldots\ldots\ldots$$

for Convergence or Divergence.

Sol. Here $\quad \Sigma a_n = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 + \ldots\ldots\ldots$

$$\Rightarrow \quad S_n = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\ S_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n(n+1)(2n+1)}{6} = \infty$$

$\Rightarrow$ $\{S_n\}$ diverges to ∞

$\Rightarrow$ Σa_n is divergent.

Example 3. *Show that the series*

$$2-2+2-2+\ldots\ldots\ldots \textit{oscillates finitely}$$

Sol. Here $\quad S_n = \begin{cases} 0, \text{if } n \text{ is even} \\ 2, \text{if } n \text{ is odd} \end{cases}$

Therefore $\underset{n\to\infty}{\text{Lt}}\ S_n$ does not exist because

$\underset{n\to\infty}{\text{Lt}}\ S_n = 0$ if n is even and 2 if n is odd.

and so the $\{S_n\}$ is not convergent.

Also $\{S_n\}$ is not divergent

and $\{S_n\}$ is bounded as Range of $\{S_n\} = \{0,2\}$ which is bounded.

$\therefore$ the sequence $\{S_n\}$ and consequently the given series oscillates finitely.

Example 4. *Show that the series*

$$\Sigma a_n = \Sigma n(-1)^{n-1} = 1 - 2 + 3 - 4 + \ldots\ldots$$

oscillates inifitely.

Sol. Here $\quad S_{2n} = 1 - 2 + 3 - 4 + \ldots\ldots\ldots + (2n-1) - 2n = -n$

and $\quad S_{2n+1} = 1 - 2 + 3 - 4 + \ldots\ldots\ldots + (2n-1) - 2n + (2n+1)$

$= n + 1$

Now $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} (-n) = -\infty$ and $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (n+1) = \infty$

Hence the sequence $\{S_n\}$ oscillates infinitely and therefore the given series Σa_n oscillates infinitely.

Example 5. *Examine the convergence of the series*

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots\ldots\ldots\ldots$$

Or

show that $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}=\frac{1}{4}$

Sol. Here $S_n = \frac{1}{1.2.3}+\frac{1}{2.3.4}+\ldots\ldots\ldots\ldots+\frac{1}{n(n+1)(n+2)}$

and $a_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right]$

(By Partial Fractions)

put $n = 1, 2, 3, \ldots\ldots\ldots\ldots, n$ we get

$$a_1 = \frac{1}{2}\left[\frac{1}{1.2}-\frac{1}{2.3}\right]$$

$$a_2 = \frac{1}{2}\left[\frac{1}{2.3}-\frac{1}{3.4}\right]$$

$$a_3 = \frac{1}{2}\left[\frac{1}{3.4}-\frac{1}{4.5}\right]$$

....

....

$$a_n = \frac{1}{2}\left[\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right]$$

Adding these n equations and cancelling the terms diagonally on R.H.S., we get

$$a_1+a_2+a_3+\ldots\ldots\ldots\ldots+a_n = \frac{1}{2}\left[\frac{1}{1.2}-\frac{1}{(n+1)(n+2)}\right]$$

$$\Rightarrow \quad S_n = \frac{1}{2}\left[\frac{1}{1.2}-\frac{1}{(n+1)(n+2)}\right]$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{2}\left[\frac{1}{1.2}-\frac{1}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1.2}-0\right]=\frac{1}{4}$$

$\Rightarrow$ $\{S_n\}$ is convergent to $\frac{1}{4}$

$\Rightarrow$ The given series is convergent and its sum is $\frac{1}{4}$

Art. 2. Preliminary Theorems.

Theorem I. *The behaviour of a series does not change on the removal, addition or alteration of a finite number of terms.*

Or

If $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two series. Suppose there exists a natural number m and an integer* $p \geq 0$ *such that* $b_n = a_{n+p}$ *for* $n > m$. *Then the two series behave alike.*

Proof. Let $\{S_n\}$ and $\{T_n\}$ be the sequences of partial sums of the series $\sum a_n$ and $\sum b_n$

For $n > m$

$$T_n = (b_1 + b_2 + \ldots\ldots + b_m) + (b_{m+1} + b_{m+2} + \ldots\ldots b_n) \qquad [\because b_n = a_{n+p} \text{ for } n > m]$$

$$= T_m + a_{m+1+p} + a_{m+2+p} + \ldots\ldots + a_{n+p}$$

$$= T_m + (S_{n+p} - S_{m+p})$$

$$= (T_m - S_{m+p}) + S_{n+p}$$

$$= C + S_{n+p}, \text{ where } C = T_m - S_{m+p} \text{ is independent of } n$$

Hence, the sequences $\{T_n\}$ and $\{S_n\}$ of the partial sums of the series $\sum b_n$ and $\sum a_n$ behave alike and consequently the two given series behave alike.

Theorem II. *The series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} K a_n, K \neq 0$, *behave alike i.e. both converge, diverge or oscillate together.*

Or

The nature of an infinite series remains unaltered if each term of the series is multiplied by non zero fixed real K.

Proof. Let $\{S_n\}$ and $\{T_n\}$ be the sequences of partial sums of the series Σa_n and $\Sigma K a_n$

$$\therefore \quad T_n = Ka_1 + Ka_2 + \ldots\ldots + Ka_n = K(a_1 + a_2 + \ldots\ldots + a_n) = K S_n$$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\, T_n = \underset{n\to\infty}{\text{Lt}}\, K S_n = K \underset{n\to\infty}{\text{Lt}}\, S_n$$

$\Rightarrow$ $\{S_n\}$ and $\{T_n\}$ behave alike

$\Rightarrow$ Σa_n and $\Sigma K a_n$, $K \neq 0$ behave alike.

Theorem III. *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two series converging respectively to A and B, then* $\sum_{n=1}^{\infty} (C_1 a_n + C_2 b_n)$ *converges to* $C_1 A + C_2 B$, *where* C_1 *and* C_2 *are real numbers.*

Proof. Let $\{S_n\}, \{T_n\}, \{S'_n\}$ be the sequences of partial sums of the series Σa_n, Σb_n and $\Sigma (C_1 a_n + C_2 b_n)$ respectively

Given Σa_n converges to A $\Rightarrow \underset{n\to\infty}{\text{Lt}}\, S_n = A$

Given Σb_n converges to B $\Rightarrow \underset{n\to\infty}{\text{Lt}}\, T_n = B$

$$S'_n = (C_1 a_1 + C_2 b_1) + (C_1 a_2 + C_2 b_2) + \ldots\ldots (C_1 a_n + C_2 b_n)$$

$$= C_1(a_1 + a_2 \ldots\ldots + a_n) + C_2(b_1 + b_2 + \ldots\ldots + b_n) = C_1 S_n +$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ S'_n = \underset{n\to\infty}{\text{Lt}}\ (C_1S_n + C_2\,T_n) = \underset{n\to\infty}{\text{Lt}}\ C_1S_n + \underset{n\to\infty}{\text{Lt}}\ C_2\,T_n$$

$$= C_1 \underset{n\to\infty}{\text{Lt}}\ S_n + C_2 \underset{n\to\infty}{\text{Lt}}\ T_n = C_1A + C_2B$$

Thus $\{S'_n\}$ converges to $C_1A + C_2B$ and consequently the series $\Sigma\,(C_1a_n + C_2\,b_n)$ converges to $C_1A + C_2B$.

Theorem IV. *If* $\sum_{n=1}^{\infty} a_n$ *converges to A and* $\{n_k\}$ *is strictly increasing sequence of natural numbers, then the series*

$$(a_1 + a_2 + \ldots\ldots\ldots\ldots + a_{n_1}) + (a_{n_1+1} + \ldots\ldots\ldots + a_{n_2}) + \ldots\ldots\ldots$$

also converges to A.

Or

The introduction of brackets in a convergent series does not affect its convergence or sum.

Proof. Let $\{S_n\}$ and $\{T_n\}$ be the sequences of nth partial sums of Σa_n and the newly formed series after the introduction of brackets.

Let the kth partial sum of the newly formed series contain n_k terms of the original series Σa_n.

Then $\quad T_k = S_{n_k}$

Now $\{S_{n_k}\}$ is a subsequence of $\{S_n\}$

$\Rightarrow\ \{T_k\}$ is a subsequence of $\{S_n\}$

and $\{S_n\}$ converges to A because Σa_n converges to A

$\Rightarrow\ \{T_k\}$ converges to A and consequently the new series converges to A.

Note. A similar result holds for divergence also.

Remark

The behaviour of a convergent or divergent series is not altered by the insertion of brackets. The same result may not hold on removal of brackets.

For example, the series

$(1-1) + (1-1) + (1-1) + \ldots\ldots\ldots\ldots$converges to zero while the series obtained after the removal of brackets is

$1-1 + 1-1+\ldots\ldots\ldots\ldots\ldots$oscillates [Prove it as in Example 3 Art. 1]

Thus convergence of given series is lost on removal of brackets.

Theorem V. *A necessary condition for convergence.*

If Σa_n *is convergent. then* $\underset{n\to\infty}{\text{Lt}}\ a_n = 0.$ *(G.N.D.U. 2003, K.U. 2000)*

Proof. Let $S_n = a_1 + a_2 + \ldots\ldots\ldots\ldots + a_{n-1} + a_n = S_{n-1} + a_n$

$$\Rightarrow \quad a_n = S_n - S_{n-1}$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ (S_n - S_{n-1}) = \underset{n\to\infty}{\text{Lt}}\ S_n - \underset{n\to\infty}{\text{Lt}}\ S_{n-1} = S - S$$

$\left[\begin{array}{l} \because \text{ given } \Sigma a_n \text{ is convergent} \\ \therefore\ \{S_n\} \text{ is convergent to S suppose} \\ \therefore\ \underset{n\to\infty}{\text{Lt}}\ S_n = S \\ \therefore\ \underset{n\to\infty}{\text{Lt}}\ S_{n-1} = S \text{ as } \{S_{n-1}\} \text{ is subsequence of } \{S_n\} \end{array}\right]$

$= 0$

which proves the required result.

Remark

The condition is not sufficient *i.e.*

$\underset{n\to\infty}{\text{Lt}}\ a_n$ may be zero without the series $\Sigma\, a_n$ being convergent.

For example, consider Σa_n, where $a_n = \dfrac{1}{\sqrt{n}}$

We shall prove that Σa_n diverges although

$$\underset{n\to\infty}{\text{Lt}}\ a_n = 0$$

Proof. $\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \dfrac{1}{\sqrt{n}} = 0$

$$\Sigma a_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots\ldots\ldots+\frac{1}{\sqrt{n}}+\ldots\ldots\ldots$$

Let
$$S_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots\ldots\ldots+\frac{1}{\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\ldots\ldots\ldots+\frac{1}{\sqrt{n}}$$

$$= \frac{n}{\sqrt{n}} = \sqrt{n} \quad \Rightarrow \quad S_n > \sqrt{n} \to \infty \text{ as } n \to \infty$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ S_n = \infty$$

$\Rightarrow$ $\{S_n\}$ diverges to ∞

$\Rightarrow$ Σa_n diverges to ∞ although $\underset{n\to\infty}{\text{Lt}}\ a_n = 0$.

Cor. *Useful form for Problems.*

$\underset{n\to\infty}{\text{Lt}}\ a_n \neq 0 \Rightarrow \Sigma a_n$ *is not convergent*

Proof. Given $\underset{n\to\infty}{\text{Lt}}\ a_n \neq 0$

If possible suppose Σa_n is convergent.

Then $\underset{n\to\infty}{\text{Lt}}\ a_n = 0$ [Theorem V]

which is contrary to (1)

So, our supposition is wrong.

Hence $\Sigma\, a_n$ is not convergent.

Remark

If $\underset{n\to\infty}{\text{Lt}}\ a_n = 0$, the nature of series cannot be concluded. Both types of convergent as well as divergent exist when $\underset{n\to\infty}{\text{Lt}}\ a_n = 0$

Example 1. *Examine the convergence of the following series :*

(*i*) $\dfrac{1}{3}+\dfrac{2}{5}+\dfrac{3}{7}+\ldots\ldots\ldots+\dfrac{n}{2n+1}+\ldots\ldots\ldots$

(*ii*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+\sqrt{n+1}}}$ (*iii*) $\displaystyle\sum_{n=1}^{\infty} 2^n$

(*iv*) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{1+3^n}$ (*v*) $1+1+1+1+\ldots\ldots\ldots$

Sol. (*i*) $\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}} \dfrac{n}{2n+1} = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{2+\frac{1}{n}} = \dfrac{1}{2} \neq 0$

$\Rightarrow$ Σa_n is not convergent (Refer Cor. Theorem V)

(*ii*) $\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{\sqrt{n}+\sqrt{n+1}} = 0$

$$a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{\sqrt{n}-\sqrt{n+1}}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n}-\sqrt{n+1}\right)}$$

$$= \frac{\sqrt{n}-\sqrt{n+1}}{\sqrt{n}-(n+1)} = \sqrt{n+1}-\sqrt{n} \qquad \text{...(1)}$$

Now $S_n = a_1 + a_2 + a_3 + ... + a_n$

$= (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + ... + (\sqrt{n+1}-\sqrt{n})$

$= \sqrt{n+1}-\sqrt{1}$

$\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\ S_n = \underset{n\to\infty}{\text{Lt}} (\sqrt{n+1}-\sqrt{1}) = \infty \Rightarrow \{S_n\}$ diverges $\Rightarrow \Sigma a_n$ diverges

(*iii*) $a_n = 2^n \Rightarrow \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ 2^n = \infty \neq 0$

$\Rightarrow$ Σa_n is not convergent

(*iv*) $a_n = \dfrac{3^n}{1+3^n} = \dfrac{1}{\left(\frac{1}{3}\right)^n + 1}$

$\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{\left(\frac{1}{3}\right)^n + 1} = \dfrac{1}{0+1} = 1 \neq 0$

$\Rightarrow$ Σa_n is not convergent.

(*v*) $a_n = 1 \Rightarrow \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ 1 \neq 0$

$\Rightarrow$ Σa_n is not convergent.

Example 2. *Prove that the Harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is not convergent where as* $\underset{n\to\infty}{\text{Lt}}\ a_n = 0$

Sol. Here $a_n = \dfrac{1}{n}$ and $\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{n} = 0$

Let $S_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + ... + \dfrac{1}{n}$

and $S_{2n} = 1 + \dfrac{1}{2} + ... + \dfrac{1}{n} + \dfrac{1}{n+1} + ... + \dfrac{1}{2n}$

Let $\epsilon = \dfrac{1}{2}$

Consider $\left|S_{2n}-S_n\right| = \left|\frac{1}{n+1}+\frac{1}{n+2}+........+\frac{1}{2n}\right| = \frac{1}{n+1}+\frac{1}{n+2}+........$

$$> \frac{1}{2n}+\frac{1}{2n}+........+\frac{1}{2n}=\frac{n}{2n}=\frac{1}{2}=\in$$

$\therefore$ $\left|S_{2n}-S_n\right| > \in \Rightarrow \{S_n\}$ is not cauchy

$\Rightarrow$ $\{S_n\}$ is not convergent.

$\Rightarrow$ $\Sigma\, a_n = \Sigma\, \frac{1}{n}$ is not convergent.

Note. $\{S_n\}$ is m, increasing and not convergent $\Rightarrow \{S_n\}$ diverges $\Rightarrow \Sigma\, a_n$ dive

Art. 3. Geometric Series

Prove that the infinite Geometric Series

$$1+r+r^2+r^3+....... = \sum_{n=1}^{\infty} r^{n-1} \text{ is}$$

(*i*) **Convergent if $|r| < 1$**
(*ii*) **Divergent if $r > 1$**
(*iii*) **Divergent If $r = 1$** } $r \geq 1$
(*iv*) **Oscillating finitely if $r = -1$**
(*v*) **Oscillating infinitely if $r < -1$**

Proof. (*i*) Let $S_n = 1+r+r^2+r^3+.......+r^{n-1} = \frac{1-r^n}{1-r} = \frac{1}{1-r}-\frac{r^n}{1-r}$ $= \frac{1}{1-r}\left[1-r^n\right]$

$$|r| < 1 \Rightarrow \underset{n\to\infty}{\text{Lt}}\, r^n = 0$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, S_n = \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{1-r}-\frac{r^n}{1-r}\right) = \frac{1}{1-r}-0 = \frac{1}{1-r}$$

$\Rightarrow$ $\{S_n\}$ converges to $\frac{1}{1-r}$

$\Rightarrow$ $\sum_{n=1}^{\infty} r^{n-1}$ converges and its sum is $\frac{1}{1-r}$

(*ii*) $r > 1 \Rightarrow \underset{n\to\infty}{\text{Lt}}\, r^n = \infty$

$$S_n = 1+r+r^2+.......+r^{n-1} = \frac{r^n-1}{r-1} = \frac{r^n}{r-1}-\frac{1}{r-1}$$

$$\underset{n\to\infty}{\text{Lt}}\, S_n = \underset{n\to\infty}{\text{Lt}}\left(\frac{r^n}{r-1}-\frac{1}{r-1}\right) = \underset{n\to\infty}{\text{Lt}}\, \frac{r^n}{r-1}-\frac{1}{r-1} = \infty$$

$\Rightarrow$ $\{S_n\}$ diverges to ∞

$\Rightarrow$ $\sum_{n=1}^{\infty} r^{n-1}$ diverges to ∞

(*iii*) $r = 1$

Now $S_n = 1+r+r^2+.......+r^{n-1} = 1+1+1+.........+1$

$= n$

$$\underset{n\to\infty}{\text{Lt}}\, S_n = \underset{n\to\infty}{\text{Lt}}\, n = \infty$$

$\Rightarrow$ $\{S_n\}$ diverges to ∞

$\Rightarrow$ $\sum_{n=1}^{\infty} r^{n-1}$ diverges to ∞

(iv) $r = -1$

$S_n = 1+r+r^2+.......+ r^{n-1} = 1-1+1-1+......+ (-1)^{n-1}$ $(\because r = -1)$

$\Rightarrow$ $S_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$

$\therefore$ $\underset{n\to\infty}{\text{Lt}}\ S_{2n} = 0$ and $\underset{n\to\infty}{\text{Lt}}\ S_{2n+1} = 1$

Now $\{S_{2n}\}$ and $\{S_{2n+1}\}$ are two subsequences of $\{S_n\}$ convergent to two different limits.

Hence $\{S_n\}$ is not convergent.

Also Range of $\{S_n\}$ is the set $\{0,1\}$ which being finite is bounded and hence $\{S_n\}$ is bounded.

We have **proved** that $\{S_n\}$ is bounded and not convergent

$\Rightarrow$ $\{S_n\}$ oscillates finitely.

$\Rightarrow$ $\sum a_n$ oscillates finitely.

(v) $r < -1$

$\Rightarrow$ r is negative but numerically greater than 1

$\Rightarrow$ Powers of r will be positive if index of r is even and negative if index is odd

$$S_n = 1+r+r^2+.......r^{n-1} = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

For $r < -1,$

$$\underset{n\to\infty}{\text{Lt}}\ S_{2n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{1-r} - \frac{r^{2n}}{1-r}\right) \frac{1}{1-r} - \infty = -\infty \qquad (\because r^{2n} \text{ is + ve})$$

and $$\underset{n\to\infty}{\text{Lt}}\ S_{2n+1} = \underset{n\to\infty}{\text{Lt}} \left[\frac{1}{1-r} - \frac{r^{2n+1}}{1-r}\right] = \frac{1}{1-r} - (-\infty) = \infty$$

Thus the sequence $\{S_n\}$ oscillates between $-\infty$ and $+\infty$.

Hence the given series $\sum_{n=1}^{\infty} r^{n-1}$ oscillates infinitely.

Note. Nature of infinite Geometric series for various values of r may be remembered s the same is useful in solving many problems.

➤ Art. 4. Cauchy's General Principal of Convergence for Series.

The series $\sum_{n=1}^{\infty} a_n$ converges if and only if, for every $\in > 0$, there exists a natural number t, such that,

$$|a_{m+1} + a_{m+2} + ... + a_n| < \in \text{ for } n > m \geq t.$$

Proof. The series $\sum_{n=1}^{\infty} a_n$ converges

Iff $\{S_n\}$ of its partial sums converges (Def.)

i.e. Iff for every $\in > 0$, $\exists$ positive integer t, such that

$$|S_n - S_m| < \in, \text{ for } n > m \geq t$$

(Cauchy criterion for convergence of sequences)

i.e. Iff $|(a_1 + a_2 + \ldots + a_m + a_{m+1} + \ldots + a_n) - (a_1 + \ldots + a_m)| < \in$, for $n > m \geq t$

i.e. Iff $|a_{m+1} + a_{m+2} + \ldots\ldots\ldots a_n| < \in$, for $n > m \geq t$.

Remark

The above theorem gives us a necessary and sufficient condition for the conv of an infinite series.

Art. 5. Series of Positive Terms

Def. A series $\sum_{n=1}^{\infty} a_n$, in which $a_n \geq 0$ for all $n \geq m$ is called a series of positiv

The words $n \geq m$ in the above definition imply that in the positive term se include all such series whose terms are positive after some particular terms bec terms before that particular term can be omitted without affecting the converg divergence of the series.

e.g. $-7-1+8+11+21+27+\ldots\ldots$ is a positive term series as its all the terms are positive after a particular term *i.e.* We shall now consider some important theorems regarding series of positive term

Theorem. *The series* $\sum_{n=1}^{\infty} a_n$ *of non-negative terms converges if and on sequence* $\{S_n\}$ *of its nth partial sums is bounded and diverges if and only if unbounded.*

Proof. Given $a_n \geq 0, \forall n$

$$\therefore \quad S_{n+1} = a_1 + a_2 + \ldots a_n + a_{n+1}$$
$$= S_n + a_{n+1} \geq S_n$$

So, $\{S_n\}$ of the series of non-negative terms is always **monotonically incre**

We know that monotone increasing sequence $\{S_n\}$ is convergent if and only bounded (above) and divergent if and only if unbounded (above) and consequen

Σa_n converges iff $\{S_n\}$ is bounded

and Σa_n diverges iff $\{S_n\}$ is unbounded.

Cor. 1. $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0 \; \forall \; n$, **either converges or diverges to ∞**

Proof. $S_{n+1} = a_1 + a_2 + \ldots + a_n + a_{n+1} = S_n + a_{n+1} \geq S_n$ $\quad [\because a$

$\Rightarrow$ $\{S_n\}$ is *m.* increasing.

Two cases arise :–

Case I. $\{S_n\}$ is bounded above

Now $\{S_n\}$ is *m.i.* and *b.* above

$\Rightarrow$ $\{S_n\}$ is convergent

$\Rightarrow$ Σa_n is convergent

Case II. $\{S_n\}$ is unbounded above

Now $\{S_n\}$ is *m.i.* and unbounded

$\Rightarrow$ $\{S_n\}$ is divergent to ∞.

$\Rightarrow$ Σa_n is divergent to ∞.

Remark

A series of non-negative terms has only two options.

It is either convergent or divergent to ∞.

In no case it can oscillate.

Cor 2. $\sum_{n=1}^{\infty} a_n, a_n \geq 0 \ \forall\ n$ and $\underset{n\to\infty}{\text{Lt}}\ a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges to ∞

Proof. $\underset{n\to\infty}{\text{Lt}}\ a_n \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is not convergent [Refer Art. 2, Theorem V, Cor.] ...(1)

$\sum_{n=1}^{\infty} a_n, a_n \geq 0 \ \forall n \Rightarrow \Sigma a_n$ is either convergent or divergent to ∞ ...(2)

From (1) and (2)

$\sum_{n=1}^{\infty} a_n$ diverges to ∞.

Remark

From Art. 5, Theorem 1, the following two important results can be concluded. See Cor. 3 and 4 below.

Cor 3. A series $\sum a_n$ of positive terms is convergent if $S_n < K$ (finite) $\forall\ n$. (Try to Prove).

Cor 4. A series $\sum a_n$ of positive terms is divergent if each term $\geq$ K (fixed positive number).

Proof. $S_n = a_1 + a_2 + \ldots\ldots + a_n \geq K + K + \ldots\ldots + K$

$\Rightarrow S_n \geq nK$ and $nK \to \infty$ as $n \to \infty$

$\underset{n\to\infty}{\text{Lt}}\ S_n = \infty \Rightarrow \{S_n\}$ is divergent to ∞

$\Rightarrow \sum a_n$ diverges to ∞.

➢ Art .6.Comparison Tests

We now proceed to find some rules with the help of which we can test the convergence or divergence of a series without computing S_n which in certain cases is quite inconvenient to find.

Comparison Tests are convenient in application. If $\sum a_n$ and $\sum b_n$ are two positive term series and we know the behaviour of one of these two series regarding convergence or divergence, then we can decide the convergence of the other series by comparing the terms of the two series.

Theorem 1. *If $\sum a_n$ and $\sum b_n$ are two series of positive terms such that $a_n \leq b_n \ \forall$ $n \geq K$ $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.*

Proof. Let $\{S_n\}$ and $\{T_n\}$ be the sequences of partial sums of the series $\sum a_n$ and $\sum b_n$

Now $S_n = a_1 + a_2 + \ldots\ldots\ldots + a_k + a_{k+1} + \ldots\ldots\ldots + a_n$

$= S_k + a_{k+1} + \ldots\ldots\ldots\ldots + a_n \leq S_k + b_{k+1} + \ldots\ldots\ldots + b_n$ $(\because a_n \leq b_n \ \forall n \geq k)$

$= S_k + (T_n - T_k)$ where $C = S_k - T_k$ is independent of n

$= (S_k - T_k) + T_n = C + T_n$ ...(1)

So, $S_n \leq C + T_n$

Given that $\sum b_n$ (positive term series) is convergent

⇒ $\{T_n\}$ is bounded above (Refer Art. 5, Theorem 1)

⇒ $\{S_n\}$ is bounded above (From (1))

⇒ $\sum a_n$ is convergent (Refer Art. 5, Theorem 1)

Cor 1. $a_n \geq 0,\ b_n \geq 0,\ a_n \leq b_n\ \forall n$ and $\sum b_n$ convergent ⇒ $\sum a_n$ converge

Cor 2. More general form of Theorem 1.

If $\sum b_n$ is a convergent series of positive terms and if $\sum a_n$ is another series of terms such that $a_n \leq h\, b_n \forall n \geq K$ where h is positive constant independent of n th is also convergent.

Proof. Let $\{S_n\}$ and $\{T_n\}$ be sequences of partial sums of the series and $\sum h\, b_n$

$$a_n \leq h\, b_n \forall n \geq K \text{ (Given)}$$

∴ $a_{k+1} \leq h\, b_{k+1},\ a_{k+2} \leq h\ b_{k+2}, \ldots\ldots\ldots\ldots$

Thus $a_{k+1} + a_{k+2} + \ldots\ldots + a_n \leq h\,(b_{k+1} + b_{k+2} + \ldots\ldots + b_n)$

⇒ $S_n - S_k \leq h\,(T_n - T_k)$

⇒ $S_n \leq h\, T_n + (S_{k-n}\, T_k)$

$\sum b_n$ is convergent and $b_n \geq 0$ (given)

⇒ $\{T_n\}$ is convergent and m-increasing

⇒ $\{T_n\}$ is converges to $l.u.b.$ say t

$$T_n \leq t \forall n$$

⇒ $h\, T_n \leq ht \forall n$ $(\because h >$

Hence from (1)

$$S_n \leq ht + (S_k - hT_k) \text{ where } S_k - hT_k = \text{finite real}$$

⇒ $\{S_n\}$ is bounded above and m. increasing

⇒ $\{S_n\}$ converges

⇒ $\sum a_n$ converges

Theorem 2. *If $\sum a_n$ and $\sum b_n$ are two positive terms series such that $a_n \geq$ K and $\sum b_n$ is divergent, then $\sum a_n$ is also divergent.*

Proof. Let $\{S_n\}$ and $\{T_n\}$ be the sequences of partial sums of the series $\sum$

Now $S_n = a_1 + a_2 + \ldots\ldots + a_k + a_{k+1} + \ldots\ldots + a_n$

$= S_k + a_{k+1} + a_{k+2} + \ldots\ldots\ldots + a_n$

$\geq S_k + b_{k+1} + b_{k+2} + \ldots\ldots\ldots + b_n$ $(\because a_n \geq$

$= S_k + T_n - T_k = (S_k - T_k) + T_n$

$= C + T_n$ where $C = S_k - T_k$ is independent of n

So, $S_n \geq C + T_n$

Given that $\sum b_n$, $b_n \geq 0$ is divergent

⇒ $\{T_n\}$ is unbounded above (Refer Art. 5, Theorem 1)

⇒ $\{S_n\}$ is unbounded above (From (1))

⇒ $\sum a_n$ is divergent (Refer Art. 5, Theorem 1)

Cor. More general form of Theorem 2.

If $\sum b_n$ is a divergent series of positive terms and if $\sum a_n$ is another series terms such that $a_n \geq h\, b_n \forall n \geq K$ where h is positive constant independent of n is also divergent.

$\sum a^n f(a^n)$ is convergent.

Thus convergence of $\sum f(n)$ implies convergence of $\sum a^n f(a^n)$.

Case 2. Let $\sum a^n f(a^n)$ be convergent

$\Rightarrow$ $\{T_n\}$ is convergent

$\Rightarrow$ $\{T_n\}$ is bounded above

$\Rightarrow$ $\{T_{n-1}\}$ is bounded above

From second part of inequality (3) it follows that $\{S_{a^n}\}$ is also bounded ab... implies that $\sum f(n)$ is convergent.

Case 3. Let $\sum f(n)$ be divergent

$\Rightarrow$ $\{S_n\}$ is divergent

$\Rightarrow$ $\{S_{a^n}\}$ is divergent $\quad$ ($\because$ $\{S_{a^n}\}$ is subsequenc...

$\Rightarrow$ $\{S_{a^n}\}$ is unbounded above

From second part of inequality (3) we get,

$$T_{n-1} \geq \frac{1}{a-1}\{S_{a^n} - af(1)\}$$

$\Rightarrow$ $\{T_{n-1}\}$ is unbounded above $\quad$ ($\because$ $\{S_{a^n}\}$ is unbound...

$\Rightarrow$ $\sum a^n f(a^n)$ is divergent

Case 4. Let $\sum a^n f(a^n)$ be divergent

$\Rightarrow$ $\{T_n\}$ is divergent

$\Rightarrow$ $\{T_n\}$ is unbounded above

From first part of inequality (3) we get,

$$S_{a^n} \geq f(1) + \frac{a-1}{a} T_n$$

$\Rightarrow$ $\{S_{a^n}\}$ is unbounded above $\quad$ ($\because$ $\{T_n\}$ is unbound...

$\Rightarrow$ $\sum f(n)$ is divergent

Hence the two series $\sum f(n)$ and $\sum a^n f(a^n)$ converge or diverge togethe...

Cor. For $a = 2$, The theorem becomes $\sum f(n)$ and $\sum 2^n f(2^n)$ behave ali...

(M.D...

Art. 8. The *p*-series

The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Also discuss the behaviour of p series for $p \leq 0$. (...

Proof. Case 1. Let $p > 0$

Let $\quad f(n) = \frac{1}{n^p}, \therefore f(n+1) = \frac{1}{(n+1)^p}$

Now $\quad \frac{1}{n^p} > \frac{1}{(n+1)^p} > 0 \quad [\because p > 0] \quad \Rightarrow \quad f(n) > f(n+1) > 0$...

$\Rightarrow$ f is positive and monotone decreasing function of n and hence b... condensation Test Cor. the series

$\sum f(n)$ and $\sum 2^n f(2^n)$ behave alike

i.e. $\quad \sum \frac{1}{n^p}$ and $\sum 2^n \left(\frac{1}{2^n}\right)^p$ behave alike $\quad$ $\left[\because f(2^n) ...\right.$

7. $\underset{n\to\infty}{\text{Lt}} \dfrac{\sin\theta}{\theta} = 1$ (calculus)

Some Standard Expansions

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\ldots.$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\ldots.$$

$$\tan^{-1}\theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7} + \ldots\ldots.$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \ldots\ldots.$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots.$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots..\infty, \text{ for } |x|<1$$

Example 1. *Prove that the series*

$$\sum a_n = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots\ldots + \frac{1}{n^n} + \ldots\ldots \textit{converges.}$$

Sol. Let $\quad \sum b_n = 1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots\ldots + \frac{1}{2^n} + \ldots\ldots$

Now $a_n \le b_n \ \forall n$

and $\sum b_n$ being Geometric Series (after omitting first term) with common ratio $\frac{1}{2} < 1$ is convergent.

Hence $\sum a_n$ converges by comparison Test. (Refer Art. 6 Theorem 1)

Example 2. *Examine the following series for convergence or divergence :*

(*i*) $\sum \frac{1}{\sqrt{n}}$ (*ii*) $\sum \frac{1}{n^p}$, $p \le 1$. (*iii*) $\sum \frac{1}{2^{n-1}+1}$ (*iv*) $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$

Sol. (*i*) $\frac{1}{\sqrt{n}} \ge \frac{1}{n} \ \forall n$ and $\sum \frac{1}{n}$ is divergent

$\Rightarrow \quad \sum \frac{1}{\sqrt{n}}$ is divergent (By Comparison Test).

(*ii*) For $p \le 1, \frac{1}{n^p} \ge \frac{1}{n} \ \forall n$ and $\sum \frac{1}{n}$ is divergent

$\Rightarrow \quad \sum \frac{1}{n^p}$ is divergent. (By Comparison Test)

(*iii*) $\frac{1}{2^{n-1}+1} < \frac{1}{2^{n-1}} = \left(\frac{1}{2}\right)^{n-1} \ \forall n$