DATE 1A/Jan/2019
PAGE

Unit 6 - 1

* **Function :-** Any Mathematical expression involving a quantity is called a function of that quantity.

Example :- Rational function, Integer functions etc.

$ax^n + bx^{n-1} + \dots + kx + l$, where n is a +ve integer, is a rational or integral function of x if x enters as a rational or integral quantity.

The Coeff. a, b, c etc can be irrational or rational or integers.

Notation :- A function of x is represented by $f(x)$, $F(x)$ or $\phi(x)$ or $P(x)$ or any other similar symbol.

* **Polynomial :-** A Polynomial is an algebraic function which is constituted of a number of terms containing different powers x connected by signs. (plus or minus)

$$\text{Polynomial} = \underset{\downarrow}{\text{Poly}} + \underset{\downarrow}{\text{nomial}}$$
many terms.

⇒ Monomial = Mono + nomial
↓ Single ↓ Term

Example :- $f(x) = 4$
$g(x) = 3x$

⇒ Binomial = Bi + nomial
↓ ↓
Two Terms

Example :-

$f(x) = 4 + 3x$.

⇒ Trinomial = Tri + nomial
↓ ↓
Three Terms.

Example :-

$f(x) = 2x^2 + x + 2$.

* Degree of a Polynomial :- Degree of a Polynomial is defined as the highest power of x, appearing in any term of a Polynomial

Example :- $3x^2 + 6x + 15x^3 + 30x^{10} + 4$.

Degree = 10.

⇒ Polynomial of degree 1 is called Linear Polynomial.

Example :- $f(x) = 4x + 3$. , $f(x) = 4x$

Degree = 1 Degree = 1

⇒ Polynomial of degree 2 is called Quadratic Polynomial.

Example :-

$ax^2 + bx + c$, where a, b, c are any constants

degree = 2

⇒ Polynomial of degree 3 is called Cubic Polynomial

Example :- $15x^3 + 10x^2 + 5x$

degree = 3.

⇒ Polynomial of degree 4 is called Biquadratic Polynomials.
Example:-

$$2x^4 - 1$$

Degree = 4

* Leading Term :- The highest power term in a polynomial is called it's leading term.

Example:-

i) $4x + 3$.

$4x$ is a leading term.

ii) $ax^2 + bx + c$

ax^2 is a leading term.

iii) $3x^2 + 6x + 15x^3 + 30x^{10} + 4$.

$30x^{10}$ is a leading term.

* Equation :- For certain value of a variable x, one given polynomial may become equal to another differently constituted polynomial. The Algebraic expression of such a relation is called an equation.

Example:-

$2x + 2 = 3x^2 + 1$ Put $x = 1$ $4 = 4$	$3x^2 - 2x - 1 = 0$

DATE
PAGE

* Root :- Any value of the quantity x, Which stisfy an equation, is called root of that equation.

* Complete Solution :- The Determination of all possible roots of an equation constitutes the complete solution of the equation.

* Complete Equation :- An Equation is said to be complete When it contains terms involving x in all its powers from n to 0 and is called In-Complete When some of the terms are absent or When some of the coefficients are zero.

Example :-

$$10x^{10} + 9x^8 + 3x^4 + 5 = 0$$

So, This is incomplete equation.

$$9x^4 + 6x^3 + 3x^2 + x + 6 = 0. \quad \Big| \quad 9x^4 + 6x^3 + 5x^2 + 4x + 3 = 0$$

So, This is Complete equation.

* General Properties of Polynomials :-

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n \quad -(1)$$

We have to find one term which effects f(x) more than others, in Polynomials.

We Can also write the above eq. (1), as

$$f(x) = a_0 x^n \Big\{ 1 + \frac{a_1}{a_0}\frac{1}{x} + \frac{a_2}{a_0}\frac{1}{x^2} + \dots + \frac{a_{n-1}}{a_0}\frac{1}{x^{n-1}} + \frac{a_n}{a_0 x^n}$$

Theorem:-

Statement:- If in the Polynomial $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$, the value of $\frac{a_k}{a_0} + 1$ or any greater value be substituted for x, where a_k is that one coefficient $a_1, a_2, a_3, \ldots, a_n$ whose numerical value is greatest, irrespective of sign, the term containing the highest power of x will exceed the sum of all the terms which follow.

Example:- $3x^3 - 2x^2 + 4x - 5$.

From theorem, $x = \frac{a_k}{a_0} + 1$ or greater value

* a_0 is the coefficient of highest power of x.

$a_0 = 3$.

* a_k is that coefficient whose numerical value is greatest

$a_k = a_3 = -5$.

$\Rightarrow x = \frac{-5}{3} + 1 = -\frac{2}{3}$

We get, $\boxed{x = -\frac{2}{3}}$

Acc. to theorem,

$3x^3 > -2x^2 + 4x - 5$

$3\left(-\frac{2}{3}\right)^3 > -2\left(-\frac{2}{3}\right)^2 + 4\left(-\frac{2}{3}\right) - 5$

$-\frac{4}{9} > -\frac{8}{9} - \frac{8}{3} - 5$

Scanned with CamScanner

$$-\frac{4}{9} > -\frac{77}{9}$$

L.H.S > R.H.S

Hence Proved.

Proof :- To Prove that

$$a_0x^n > a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n \text{ for Certain values of } x. \quad (*)$$

The inequality (*) is satisfied by any value of x for Which

$$a_0x^n > a_k(x^{n-1} + x^{n-2} + \dots + x + 1) \quad \text{—} (1)$$

where a_k is the greatest Coff. among $a_1, a_2, a_3, \dots, a_n$ without regard to sign.

for understanding

if x satisfies (1), then it satisfies *

i.e. $a_0x^n > a_k\frac{(x^n-1)}{(x-1)}$ | $a_0x^n > a_k\left(\frac{1-x^n}{1-x}\right)$

Sum of G.P.
when $r>1$: $S_n = \frac{a(r^n-1)}{r-1}$ | $r<1$: $S_n = \frac{a(1-r^n)}{1-r}$

$x^n > \frac{a_k}{a_0}\left(\frac{x^n-1}{x-1}\right)$ | $x^n > \frac{a_k}{a_0}\left(\frac{1-x^n}{1-x}\right)$

By taking negative sign Common.

$\Rightarrow x^n > \frac{a_k}{a_0(x-1)}(x^n-1)$ | $x^n > \frac{a_k}{a_0(x-1)}(x^n-1)$

i.e. $x^n > \frac{a_k}{a_0(x-1)}(x^n-1)$

$\Rightarrow x^n > \lambda(x^n-1)$ where $\lambda = \frac{a_k}{a_0(x-1)}$

$\Rightarrow \lambda \leq 1.$

$\Rightarrow \frac{a_k}{a_0(x-1)} \leq 1.$

$\Rightarrow \frac{a_k}{a_0} \leq (x-1).$

$\Rightarrow \frac{a_k}{a_0} + 1 \leq x.$

$\Rightarrow \boxed{x \geq \frac{a_k}{a_0} + 1.}$ Hence Proved.

Theorem 2 :-

Statement : If in the polynomial $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$, the value of $\frac{a_n}{a_n + a_k}$ or any other smaller value be substituted for x, where a_k is the greatest cofficient except of a_n term, a_n will be numerically greater than the sum of all others.

Example : $3x^3 - 2x^2 + 4x + 5.$

Put $x = \frac{a_n}{a_n + a_k} = \frac{5}{10} = \frac{1}{2}.$ $\quad a_n = 5$, $a_k = 5.$

$5 > 3x^3 - 2x^2 + 4x$. (Acc. to this theorem)

$5 > 3\left(\frac{1}{8}\right) - 2\left(\frac{1}{4}\right) + 2$

$5 > \frac{3}{8} - \frac{1}{2} + 2$

$$5 > \frac{3-4+16}{8}$$

$$5 > \frac{15}{8}$$

$5 > 1.875$, Hence Proved

<u>Proof</u>:- Given Polynomial is

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

Put $x = \frac{1}{y}$.

then $f\left(\frac{1}{y}\right) = a_0\frac{1}{y^n} + a_1\frac{1}{y^{n-1}} + \cdots + a_{n-1}\frac{1}{y} + a_n.$

$$f\left(\frac{1}{y}\right) = \frac{1}{y^n}\left[a_0 + a_1y + a_2y^2 + \cdots + a_{n-1}y^{n-1} + a_ny^n\right]$$

$$y^n f\left(\frac{1}{y}\right) = a_0 + a_1y + a_2y^2 + \cdots + a_{n-1}y^{n-1} + a_ny^n.$$

$= g(y)$ say.

also, $a_k > a_0$, $a_k > a_1$, $a_k > a_2$, $\cdots$, $a_k > a_{n-1}$

$$g(y) = a_0 + a_1y + a_2y^2 + \cdots + a_{n-1}y^{n-1} + a_ny^n. \quad (*)$$

By previous theorem.

$$a_ny^n > a_{n-1}y^{n-1} + a_{n-2}y^{n-2} + \cdots + a_1y + a_0 \quad (*)$$

if $y \geq \frac{a_k}{a_n} + 1$

DATE
PAGE

from eq. (xi)

$\Rightarrow a_n y^n > y^n\left(\frac{a_{n-1}}{y} + \frac{a_{n-2}}{y^2} + \cdots + \frac{a_1}{y^{n-1}} + \frac{a_0}{y^n}\right)$

$\Rightarrow a_n > a_{n-1}\frac{1}{y} + a_{n-2}\frac{1}{y^2} + \cdots + a_1\frac{1}{y^{n-1}} + a_0\frac{1}{y^n}$

$\Rightarrow a_n > a_{n-1}x + a_{n-2}x^2 + \cdots + a_1x^{n-1} + a_0x^n.$

if $\frac{1}{x} \geq \frac{a_k + a_n}{a_n}$

$\Rightarrow a_n > a_{n-1}x + a_{n-2}x^2 + \cdots + a_1x^{n-1} + a_0x^n.$

if $x \leq \frac{a_n}{a_k + a_n}$, Hence Proved.

* Change of form of a Polynomial Corresponding to an increase or decrease (diminution) of the variable.

Example :-

$$f(x) = 10x^3 + 13x^2 + 14x - 7.$$

When $x = 1$, $f(1) = 10 + 13 + 14 - 7 = 30$

$x = 0.9$.

$f(0.9) = 10(0.9)^3 + 13(0.9)^2 + 14(0.9) - 7.$

$(x \pm 1)$
$x = 0.9$
$(x - 0.1)$.
$(x +$

$x = 1.2$

$f(1.2) = 10(1.2)^3 + 13(1.2)^2 + 14(1.2) - 7.$

By Calculationg $f(0.9)$ & $f(1.2)$, we can find the difference/Change of Polynomial.

i.e. We have to see What is the difference in Polynomial with the change of variable?

:- Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$ be given Polynomial.

Let $h > 0$.

$$f(x+h) = a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \dots + a_{n-1}(x+h) + a_n$$

$$f(x+h) = a_0\left[{}^nC_0x^n + {}^nC_1x^{n-1}h^1 + {}^nC_2x^{n-2}h^2 + \dots + {}^nC_{n-1}xh^{n-1} + {}^nC_nx^0h^n\right]$$

$$a_1\left[{}^{n-1}C_0x^{n-1} + {}^{n-1}C_1x^{n-2}h + {}^{n-1}C_2x^{n-3}h^2 + \dots + {}^{n-1}C_{n-2}xh^{n-2} + {}^{n-1}C_{n-1}x^0h^{n-1}\right]$$

$$a_2\left[{}^{n-2}C_0x^{n-2} + {}^{n-2}C_1x^{n-3}h + {}^{n-2}C_2x^{n-4}h^2 + \dots + {}^{n-2}C_{n-3}xh^{n-3} + {}^{n-2}C_{n-2}x^0h^{n-2}\right.$$

$$f(x+h) = \overset{a_{n-2}(x+h)^2}{\wedge} + a_{n-1}(x+h) + a_n.$$

$$f(x+h) = a_0\left[x^n + nx^{n-1}h^1 + \frac{n(n-1)}{2!}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n\right]$$

$$+ a_1\left[x^{n-1} + (n-1)x^{n-2}h + \frac{(n-1)(n-2)}{2!}x^{n-3}h^2 + \dots + h^{n-1}\right]$$

$$+ a_2\left[x^{n-2} + (n-2)x^{n-3}h + \frac{(n-2)(n-3)}{2!}x^{n-4}h^2 + \dots + h^{n-2}\right]$$

$$\dots + a_{n-2}(x^2+2hx+h^2) + a_{n-1}(x+h) + a_n.$$

$$f(x+h) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$
$$+ h\left[na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \dots + a_{n-1}\right]$$
$$+ \frac{h^2}{2!}\left[n(n-1)a_0x^{n-2} + (n-1)(n-2)a_1x^{n-3} + (n-2)(n-3)a_2x^{n-4} + \dots 2a_{n-2}\right]$$
$$\vdots$$
$$+ \dots + \frac{h^n}{n!}(a_0 n!)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots + \frac{h^n}{n!}f^n(x).$$

Interchange x & h.

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \dots + \frac{x^n}{n!}f^n(h)$$

Example:- $f(x) = 4x^3 + 6x^2 + 4 - 7x.$

find $f(x+1)$.

$$\therefore f(x+1) = f(1) + xf'(1) + \frac{x^2}{2!}f''(1) + \frac{x^3}{3!}f'''(1).$$

$f(1) = 4(1)^3 + 6(1)^2 - 7(1) + 4$
$= 4 + 6 - 7 + 4$
$= 7.$

$f'(x) = 12x^2 + 12x - 7.$
$f'(1) = 12 + 12 - 7$
$= 17.$

$f''(x) = 24x + 12.$
$f''(1) = 24 + 12$
$= 36$

$f'''(x) = 24$
$f'''(1) = 24.$

$$f(x+1) = 7 + x(17) + \frac{x^2(36)}{2} + \frac{x^3 24}{6}.$$

$= 7 + 17x + 18x^2 + 4x^3.$
$f(x+1) = 4x^3 + 18x^2 + 17x + 7.$

⇒ Verify $f(x+1)$ in direct way.
$f(x) = 4x^3 + 6x^2 + 4 - 7x.$
$f(x+1) = 4(x+1)^3 + 6(x+1)^2 - 7(x+1) + 4.$

$= 4[x^3 + 1 + 3x^2 + 3x] + 6[x^2 + 2x + 1] - 7x - 7 + 4.$
$= 4x^3 + 4 + 12x^2 + 12x + 6x^2 + 12x + 6 - 7x - 7 + 4$
$= 4x^3 + 18x^2 + 17x + 7.$

② $f(x) = 3x^3 - 2x^2 + 4x - 5$
find $f(x+2)$.

$f(2) = 3(2)^3 - 2(2)^2 + 4(2) - 5$
$= 24 - 8 + 8 - 5$
$= \cancel{28 - 13}. \quad 24 - 5$
$\cancel{= 15} \quad = 19.$

$f'(x) = 9x^2 - 4x + 4$
$f'(2) = 9(2)^2 - 4(2) + 4$
$= 36 - 8 + 4$
$= 40 - 8$
$= 32.$

$f''(x) = 18x - 4.$
$f''(2) = 18(2) - 4$
$= 36 - 4$
$= 32.$

$f'''(x) = 18$
$f'''(2) = 18$

$$f(x+2) = f(2) + x f'(2) + \frac{x^2}{2!} f''(2) + \frac{x^3}{3!} f'''(2).$$

$= 19 + 32x + 16x^2 + 3x^3$
$f(x+2) = 3x^3 + 16x^2 + 32x + 19.$

⇒ Verify $f(x+2)$ in direct way.
$f(x+2) = 3(x+2)^3 - 2(x+2)^2 + 4(x+2) - 5$
$= 3[x^3 + 8 + 6x^2 + 12x] - 2[x^2 + 4 + 4x] + 4x + 8 - 5$
$= 3x^3 + 24 + 18x^2 + 36x - 2x^2 - 8 - 8x + 4x + 3$
$= 3x^3 + 16x^2 + 32x + 19.$

* Form of the Quotient and Remainder When a Polynomial is divided by a binomial.

∴ Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$ be divided by $x-h$. and Q & R. be the quotient & remainder respectively.

Then,

Dividend = Divisor × Quotient + Remainder.

$f(x) = (x-h)\,Q + R$ — (*).

Let $Q = b_0x^{n-1} + b_1x^{n-2} + \dots + b_{n-2}x + b_{n-1}$

⇒ We let Q of degree $(n-1)$ because In L.H.S $f(x)$ has degree (n).
So, in R.H.S we also want degree n
as we know $(x-h)$ has 1 degree
That's why we take Q of degree $(n-1)$.

Put Q in (*).

$$f(x) = (x-h)[b_0x^{n-1} + b_1x^{n-2} + \dots + b_{n-2}x + b_{n-1}] + R$$
$$= b_0x^n + (b_1 - hb_0)x^{n-1} + (b_2 - hb_1)x^{n-2} + (b_3 - hb_2)x^{n-3} + \dots + (b_{n-1} - hb_{n-2})x + (-hb_{n-1}) + R$$

$$f(x) = b_0x^n + (b_1 - hb_0)x^{n-1} + (b_2 - hb_1)x^{n-2} + \dots + (b_{n-1} - hb_{n-2})x + (R - hb_{n-1}).$$

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = b_0x^n + (b_1 - hb_0)x^{n-1} + (b_2 - hb_1)x^{n-2} + \dots + (b_{n-1} - hb_{n-2})x + (R - hb_{n-1})$$

Equating co-efficients of same powers of x on both side we get.

$a_0 = b_0 \Rightarrow b_0 = a_0$

$a_1 = b_1 - hb_0 \Rightarrow b_1 = a_1 + hb_0 = a_1 + ha_0$

$a_2 = b_2 - hb_1 \Rightarrow b_2 = a_2 + hb_1 = a_2 + h(a_1 + ha_0)$

$\vdots$

$a_{n-1} = b_{n-1} - hb_{n-2} \Rightarrow b_{n-1} = a_{n-1} + hb_{n-2}$

$a_n = \cancel{b_n} R - hb_{n-1} \Rightarrow \cancel{b_n} R = a_n + hb_{n-1}$

Once Again,

Equating Co-efficients of same powers of x on both side we get

$a_0 = b_0 \Rightarrow b_0 = a_0$

$a_1 = b_1 - hb_0 \Rightarrow b_1 = a_1 + hb_0 = a_1 + ha_0$

$a_2 = b_2 - hb_1 \Rightarrow b_2 = a_2 + hb_1 = a_2 + h(a_1 + ha_0)$

$\vdots$

$a_{n-1} = b_{n-1} - hb_{n-2} \Rightarrow b_{n-1} = a_{n-1} + hb_{n-2}$

$a_n = R - hb_{n-1} \Rightarrow R = a_n + hb_{n-1}$

$\Rightarrow Q = a_0x^{n-1} + (a_1 + hb_0)x^{n-2} + (a_2 + hb_1)x^{n-3} + \dots + (a_{n-1} + hb_{n-2})$

$R = a_n + hb_{n-1}$

Synthetic Division:-

h	a_0	a_1	a_2	a_3	$\dots$	a_{n-1}	a_n
		ha_0	$h(a_1+ha_0)$	hb_2		hb_{n-2}	hb_{n-1}
	a_0	a_1+ha_0	a_2+hb_1	a_3+hb_2	$\dots$	$a_{n-1}+hb_{n-2}$	a_n+hb_{n-1}
	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$		$\Downarrow$	$\Downarrow$
	b_0	b_1	b_2	b_3		b_{n-1}	R

Note:-

$Q(x) = b_0x^{n-1} + b_1x^{n-2} + b_2x^{n-3} + \dots + b_{n-3}x^2 + b_{n-2}x + b_{n-1}$

$R(x) = a_n + hb_{n-1}$

Example:- Find the Quotient and Remainder When $3x^4 - 5x^3 + 10x^2 + 11x - 61$ is divided by $x - 3$.

	x^4	x^3	x^2	x	c
3	3	-5	10	11	-61
	$\downarrow$	9	12	66	231
	3	4	22	77	170

Scanned with CamScanner

$Q = 3x^3 + 4x^2 + 22x + 77.$

$R = 170$

Hw. Exercise :-

Find Quotient and Remainder, When verify

i) $x^3 + 5x^2 + 3x + 2$ is divided by $x-1$ also find $f(1)$. $[f(1) = R]$

ii) $x^5 - 4x^4 + 7x^3 - 11x - 13$ is divided by $x-5$ also find $f(5)$. Verify $f(5) = R$ of $f(x)$.

iii) $x^9 + 3x^7 - 15x^2 + 2$ is divided by $x-2$ also find $f(2)$. Verify $f(2) = R$ of above $f(x)$.

iv) $x^5 + x^2 - 10x + 113$ is divided by $x+4$ also find $f(-4)$. Verify $f(-4) = R$ of $f(x)$.

Note :-

$f(x) = (x-h)Q(x) + R$

Put $x = h$.

$f(h) = R.$

⇒ Above article tells us a convenient way to evaluate/calculate the numerical value of a polynomial when any no. is substituted for x.

From example $f(x) = 3x^4 - 5x^3 + 10x^2 + 11x - 61.$

find $f(3)$.

$^2 + 11x - 61$

$f(3) = 3(3)^4 - 5(3)^3 + 10(3)^2 + 11x - 61.$

$f(x) = (x-3)Q(x) + R.$

$f(3) = (3-3)Q(x) + R$

$f(3) = R.$

$f(3) = 170$

Homework.

1) Find the Quotient and Remainder of x^3+5x^2+3x+2 is divided by $x-1$ and also find $f(1)$ & Verify $f(1)$ = Remainder

:-

	x^3	x^2	x	c
1	1	5	3	2
	↓	1	6	9
	1	6	9	11

$Q = x^2+6x+9$

$f(1) = R = 11$

$f(x) = x^3+5x^2+3x+2$

$f(1) = (1)^3+5(1)^2+3(1)+2$

$f(1) = 1+5+3+2$

$f(1) = 11$

$f(1) = R$ $[11 = R]$.

Hence Verified.

ii) $f(x) = x^5-4x^4+7x^3-11x-13$ is divided by $x-5$. and also find $f(5)$ & verify $f(5)$ = Remainder.

:-

	x^5	x^4	x^3	x^2	x	c
5	1	-4	7	0	-11	-13
		5	5	60	300	1445
	1	1	12	60	289	$\boxed{1432}$

$Q = x^4+x^3+12x^2+60x+289$

$f(5) = R = 1432$.

$f(x) = x^5 - 4x^4 + 7x^3 - 11x - 13.$

$f(5) = (5)^5 - 4(5)^4 + 7(5)^3 - 11(5) - 13.$

$f(5) = 3125 - 4(625) + 7(125) - 11(5) - 13$

$f(5) = 3125 - 2500 + 875 - 55 - 13.$

$f(5) = 1432$

$f(5) = R.$

Hence Verified.

iii) Find Quotient & Remainder of $f(x) = x^9 + 3x^7 - 15x^2 + 2$ is divided by $x-2$ and also find $f(2)$ & verify $f(2)$ = remainder

:-

	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	c
2	1	0	3	0	0	0	0	-15	0	2
		2	4	14	28	56	112	224	418	836
	1	2	7	14	28	56	112	209	418	838.

$Q = x^8 + 2x^7 + 7x^6 + 14x^5 + 28x^4 + 56x^3 + 112x^2 + 209x + 418$

$R = 838$

$f(x) = x^9 + 3x^7 - 15x^2 + 2.$

$f(2) = (2)^9 + 384 - 60 + 2$

$= 512 + 384 - 60 + 2$

$f(2) = 838$

$f(2) = R$

Hence verified.

iv) $f(x) = x^5 + x^2 - 10x + 113$ is divided by $x+4$ and also find $f(-4)$. & Verify $f(-4)$ = Remainder.

	x^5	x^4	x^3	x^2	x	C
-4	1	0	0	1	-10	113.
		-4	16	-64	252	-968
	1	-4	16	-63	242	-855

$Q = x^4 - 4x^3 + 16x^2 - 63x + 242.$

$R = -855$

$f(x) = \cdot x^5 + x^2 - 10x + 113$

$f(-4) = (-4)^5 + (4)^2 - 10(4) + 113$

$f(-4) = -1025 + 16 + 40 + 113$

$f(-4) = -855.$

$f(-4) = R.$

Hence verified.

Euclid's Division Algorithm:

Let $f(x)$ and $g(x)$ be two polynomials, where $g(x)$ is non-zero $[g(x) \neq 0]$. Then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\boxed{f(x) = g(x)\,q(x) + r(x)}$$; where the degree of $r(x)$ $= 0$ or degree of $r(x) <$ degree of $g(x)$. or $r(x) = 0$.

$\Rightarrow$ Where the either $r(x) = 0$ or $0 \leq \deg r(x) < \deg g(x)$.

Example:

* $f(x) = x^2 + 2x + 1$

$g(x) = x - 1$.

$$
\begin{array}{r|l}
x-1 & x^2 + 2x + 1 \quad \big(x + 3 \\
 & \underline{-x^2 \mp x} \\
 & \quad 3x + 1 \\
 & \quad \underline{\overset{-}{3x} \overset{+}{-3}} \\
 & \qquad 4
\end{array}
$$

$4 \rightarrow r(x) \rightarrow \deg[r(x)] = 0$

$\deg g(x) = 1$.

$\Rightarrow \deg r(x) < \deg g(x)$.

* $f(x) = x^4 + x + 1$

$g(x) = x^3$.

$\deg g(x) = 3$

$$
\begin{array}{r|l}
x^3 & x^4 + x + 1 \quad \big(x \\
 & \underline{x^4} \\
 & \quad x + 1
\end{array}
$$

$x + 1 \rightarrow r(x)$.

$\deg r(x) = 1$.

$\Rightarrow \deg r(x) < \deg g(x)$.

Proof :-

Case 1 :-

$f(x) = 0$

$f(x) = g(x).0 + 0$

$\quad = g(x).q(x) + r(x)$

So here, $r(x) = 0$

Case 2 :-

$f(x) \neq 0$

We will apply Mathematical Induction on deg of $f(x)$

Let $\deg[f(x)] = 0$

$\Rightarrow f(x) = a$, where a is constant

Subcase 1 :- If $\deg g(x) = 0$

$\Rightarrow g(x) = b$, where $b \neq 0$ is constant.

then, $a = \frac{a}{b}.b + 0 \Rightarrow a = b.\frac{a}{b} + 0$

$\Rightarrow f(x) = g(x).q(x) + r(x)$, where $r(x) = 0$

$q(x) = \frac{a}{b}$

$f(x) = g(x).q(x)$

$\frac{f(x)}{g(x)} = q(x)$

$\frac{a}{b} = q(x)$

Subcase 2 :- If $\deg g(x) \neq 0$

$\Rightarrow \deg g(x) \geq 1$.

$(x+1)$

$\boxed{\deg \geq 1}$

0

a

0

$a \to r(x)$

then $a = g(x).0 + a$.

$f(x) = g(x).q(x) + r(x)$, where $r(x) = a$

$q(x) = 0$

$\deg r(x) = 0 < 1 \leq$ degree of $g(x)$.

$\Rightarrow \deg r(x) < \deg g(x)$.

$\Rightarrow$ Euclid's Division algorithm is true if degree $f(x) = 0$

Let us assume that the algorithm is true for those polynomials whose degree is less than k.

Case 3:- We will prove that the algorithm is true for the polynomial whose degree is k.

Let $\deg f(x) = k$

Suppose $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_{k-1}x^{k-1} + a_kx^k$
where $a_k \neq 0$.

Let $g(x) = b_0 + b_1x + b_2x^2 + \dots + b_rx^r$,
where $b_r \neq 0$

Subcase 1:- If $r > k$.
$\deg g(x) > \deg f(x)$.

$f(x) = g(x).0 + f(x)$

$f(x) = g(x).q(x) + r(x)$, where $r(x) = f(x)$.

$\therefore \deg r(x) = k$.
or $k < r$ [r is $\deg g(x)$]
$\Rightarrow \deg r(x) < \deg g(x)$.

$f(x) = x^2+2x+1$
$g(x) = x^4+2x$

$$x^4+2x \,\overline{)\, x^2+2x+1 \,}\,(\, 0$$
$$\frac{0}{x^2+2x+1}$$

Subcase 2:- If $r \le k$.

Let $f_1(x) = f(x) - \frac{a_k}{b_r}x^{k-r}g(x)$.

$f(x) = x^2+2x+1$
$g(x) = x^2$ or $x+1$

$= a_0 + a_1x + \dots + a_{k-1}x^{k-1} + a_kx^k - \frac{a_k}{b_r}x^{k-r}[b_0 + b_1x + \dots + b_rx^r]$

$$= \left[a_0 + a_1 x + a_2 x^2 + \dots + a_{k-1} x^{k-1} + a_k x^k\right] - \left[\frac{a_k b_0}{b_r} x^{k-r} + \frac{a_k b_1}{b_r} x^{k-r+1} + \right.$$

= Polynomial of degree less than k.

So, By Induction process result is true for $f_1(x)$.

$$\Rightarrow f_1(x) = g(x)\, q(x) + r(x), \text{ where } r(x) = 0 \text{ or }$$
$$0 \le \deg r(x) < \deg g(x).$$

$$\Rightarrow f(x) - \frac{a_k}{b_r} x^{k-r} g(x) = g(x)\, q(x) + r(x).$$

$$\Rightarrow f(x) = g(x)\left[q(x) + \frac{a_k}{b_r} x^{k-r}\right] + r(x), \text{ where } r(x) = 0 \text{ or}$$
$$0 \le \deg r(x) < \deg g(x)$$

$$\Rightarrow f(x) = g(x)\, q_1(x) + r(x), \text{ where } r(x) = 0 \quad \text{or}$$
$$0 \le \deg r(x) < \deg g(x)$$

Hence proved.
Hence result is true for polynomials of degree k.
So, By Induction process, the result is true for all the polynomials

$^{k+1} + \cdots + a_k x^k \Big]$

Uniqueness:-

Assume that there exist $q(x)$, $r(x)$, $q_1(x)$ & $r_1(x)$ such that

$$f(x) = g(x)q(x) + r(x); \quad r(x)=0 \text{ or } 0 \le \deg r(x) < \deg g(x) \quad \text{①}$$

$$\text{and} \quad f(x) = g(x)q_1(x) + r_1(x); \quad r_1(x)=0 \text{ or } 0 \le \deg r_1(x) < \deg g(x) \quad \text{②}$$

① − ②

$$0 = g(x)\left[q(x) - q_1(x)\right] + r(x) - r_1(x)$$

$$\Rightarrow r_1(x) - r(x) = g(x)\left[q(x) - q_1(x)\right] \quad \text{— ③}$$

as $q(x) \neq q_1(x) \Rightarrow q(x) - q_1(x) \neq 0$

$$\Rightarrow \deg\left[q(x) - q_1(x)\right] \ge 0 \quad \text{— } (*)$$

also, $\deg\left(g(x)\left[q(x) - q_1(x)\right]\right) = \deg g(x) + \deg\left[q(x) - q_1(x)\right]$

$\deg\left(g(x)\left[q(x) - q_1(x)\right]\right) \ge \deg g(x) + 0 \quad -(**)$ [using $(*)$]

From eq. ③

$$\deg\left[r_1(x) - r(x)\right] = \deg\left(g(x)\left[q(x) - q_1(x)\right]\right)$$

$$\Rightarrow \deg\left[r_1(x) - r(x)\right] \ge \deg g(x) + 0 \quad \text{— ④} \quad [\text{using}(**)]$$

but $\deg r(x) < \deg g(x)$

or $\deg r_1(x) < \deg g(x)$

$r(x) = 2x+3$
$r_1(x) = 4x^2+4$
$g(x) = 8x^3+3x^2+9$
$r_1(x) - r(x) = 4x^2 - 2x + 1$

$\Rightarrow \deg[r_1(x) - r(x)] < \deg g(x)$

Which is a contradiction to our conclusion (4)

$\Rightarrow q(x) = q_1(x)$
then from eq. (3)

$$r_1(x) - r(x) = g(x)(0)$$
$$r_1(x) - r(x) = 0$$
$$r_1(x) = r(x)$$

hence Uniqueness is proved.

Examples:-

(1) If $ax^3 + bx^2 + x - 6$ has $x+2$ as a factor and leaves remainder 4 when divided by $x-2$, then find a and b.

:- $f(x) = ax^3 + bx^2 + x - 6$.
as $x+2$ is a factor of $f(x)$.
$\Rightarrow x+2$ divides $f(x)$ and leaves remainder zero.
$\Rightarrow f(-2) = 0$ — (1)
$f(-2) = a(-2)^3 + b(-2)^2 + (-2) - 6$
from eq. (1)
$\Rightarrow -8a + 4b - 8 = 0$
$-8a + 4b = 8$ — (2)

$\Rightarrow x-2$ divides $f(x)$ and leaves remainder 4. $f(x)$ $(x-2)$
$f(2) = 4$
$a(2)^3 + b(2)^2 + 2 - 6 = 4$
$8a + 4b - 4 = 4$
$8a + 4b = 8$ — (3)

DATE
PAGE

Add eq. (2) & (3)

$-8a + 4b = 8$

$8a + 4b = 8$

$8b = 16$

$\boxed{b = 2}$

Put b in eq. (3)

$8a + 4(2) = 8$

$8a + 8 = 8$

$8a = 0$

$\boxed{a = 0}$

(2) Prove that when the polynomial $f(x)$ is divided by $(x-\alpha)(x-\beta)$; $\alpha \neq \beta$; then the remainder is

$$\frac{(x-\beta)f(\alpha) - (x-\alpha)f(\beta)}{\alpha - \beta}.$$

Example:- $f(x) = x^4 + x^2 + 1$

$g(x) = (x-\alpha)(x-\beta)$, where $\alpha = 2$

$= (x-2)(x-1)$. $\beta = 1$.

$$\text{Remainder} = \frac{(x-\beta)f(\alpha) - (x-\alpha)f(\beta)}{\alpha - \beta}$$

Put α & β

$$R = \frac{(x-1)f(2) - (x-2)f(1)}{2-1}$$

$= (x-1)(16-4+1) - (x-2)(1+1+1)$

$= 21(x-1) - 3(x-2)$

$= 21x - 21 - 3x + 6$

$= 18x - 15$

Prove of Question (2)

Sol: let $(x-\alpha)(x-\beta) = g(x) \Rightarrow \deg(g(x)) = 2.$

Then by Euclid's Division Algorithm.

$f(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $0 \le \deg r(x) < \deg g(x)$

$\Rightarrow f(x) = (x-\alpha)(x-\beta)\, q(x) + r(x)$, where $r(x) = 0$ or $0 \le \deg r(x) < 2$ — (1)

$\boxed{\deg g(x) = 2}$

as $\deg r(x) < 2$

$\Rightarrow r(x) = Ax + B$, where A and B are constants

$\Rightarrow f(x) = (x-\alpha)(x-\beta)q(x) + Ax + B$ — (2)

Put $x = \alpha$ in (2)

$f(\alpha) = A\alpha + B$ — (3)

Put $x = \beta$ in (2)

$f(\beta) = A\beta + B$ — (4)

(3) − (4)

$f(\alpha) - f(\beta) = A(\alpha - \beta)$

$$\Rightarrow \boxed{A = \frac{f(\alpha) - f(\beta)}{\alpha - \beta}}$$

Put Value of A in (3)

$$f(\alpha) = \left[\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right]\alpha + B$$

$$B = f(\alpha) - \frac{[f(\alpha) - f(\beta)]}{\alpha - \beta}\,\alpha$$

$$= \frac{\alpha \cancel{f(\alpha)} - \beta f(\alpha) - \cancel{\alpha f(\alpha)} + \alpha f(\beta)}{\alpha - \beta}$$

g(x).

$$\cancel{B = \frac{(\alpha - \beta) f(\beta)}{\alpha - \beta}}$$

$$\boxed{B = \frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta}}$$

We know,

$$r(x) = Ax + B \qquad [\text{Put } A \,\&\, B]$$

$$= \frac{[f(\alpha) - f(\beta)]x}{\alpha - \beta} + \frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta}$$

$$= \frac{f(\alpha)x - f(\beta)x + \alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta}$$

$$\boxed{r(x) = \frac{f(\alpha)(x - \beta) - f(\beta)(x - \alpha)}{\alpha - \beta}}$$ Hence Proved.

GCD (Greatest Common Divisor):-

Let a and b be two numbers, then d is called gcd of a & b if.

① d/a and d/b $\left[\frac{a}{d} \text{ and } \frac{b}{d}\right]$

② If there is some number c s.t. c/a and c/b then $\frac{a}{c}, \frac{b}{c}$

$c \leq d$.

* GCD of Two Polynomials:-

Let $f(x)$ and $g(x)$ be two polynomials. Then a polynomial $h(x)$ is called gcd of $f(x)$ & $g(x)$.

① $h(x)/f(x)$ & $h(x)/g(x)$

② If $r(x)|f(x)$ and $r(x)|g(x)$ then $r(x)|h(x)$.

Example:- (i) $f(x) = x+1$ $\quad g(x) = x^2-1 = (x+1)(x-1)$.

let $h(x) = x+1$

$h(x)/f(x)$ & $h(x)/g(x)$.

$gcd = x+1$

② $f(x) = x^2-1$
$= (x-1)(x+1)$
$g(x) = x^2+1$
$= (x+i)(x-i)$

$cd = 1$

1 is the only common divisor of $f(x)$ & $g(x)$

hence gcd. $(x^2-1, x^2+1) = 1$

(3) $f(x) = x^2 - 1$, $g(x) = x^3 + x^2 - x - 1$
$= (x-1)(x+1)$
$= x^2(x+1) - 1(x+1)$
$= (x^2-1)(x+1)$
$= (x-1)(x+1)(x+1)$

Cd of $f(x)$ & $g(x) = 1, x-1, x+1, (x-1)(x+1)$
gcd of $f(x)$ & $g(x) = (x-1)(x+1)$
$= x^2 - 1.$

Theorem:- If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct roots of a polynomial equation. Then $(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n) / f(x)$

Proof:- We will prove the theorem by PMI on n.
When $n=1$ i.e. When α_1 is the root of $f(x)$.

T.P. $(x-\alpha_1)/f(x)$.

By Euclid's division Algorithm on $f(x)$ and $x - \alpha_1$
$f(x) = (x-\alpha_1)q(x) + r(x)$; where $r(x) = 0$ or — (1)
$0 \le \deg r(x) < \deg(x-\alpha_1)$
$\Rightarrow 0 \le \deg r(x) < 1.$

Put $x = \alpha_1$

$f(\alpha_1) = 0 + r(\alpha_1).$ — (2)

also α_1 is root of $f(x) \Rightarrow f(\alpha_1) = 0$

so from (2) $r(\alpha_1) = 0$ — (*)
also $\deg r(x) = 0$
$\Rightarrow r(x) = A \Rightarrow r(\alpha_1) = A$

from (*).

$\Rightarrow 0 = A$ $\quad\quad$ $[r(\alpha_1) = 0]$

So from eq ①

$f(x) = (x-\alpha_1)q(x)$

$\Rightarrow (x-\alpha_1) \mid f(x)$.

hence result is true when $n=1$.

$\Rightarrow$ Assume that the result is true for $n=k-1$

i.e. if $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{k-1}$ are roots of $f(x)$, then

$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_{k-1}) \mid f(x)$ — (**)

When, $n=k$

i.e., If $\alpha_1, \alpha_2, \dots, \alpha_k$ are roots of $f(x)$.

T.P $(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{k-1})(x-\alpha_k) \mid f(x)$

by statement (**).

$f(x) = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{k-1})\,q_1(x)$ - ③

Put $x = \alpha_k$

$f(\alpha_k) = (\alpha_k-\alpha_1)(\alpha_k-\alpha_2)\dots(\alpha_k-\alpha_{k-1})\,q_1(\alpha_k)$.

also, $f(\alpha_k) = 0$

$\Rightarrow 0 = (\alpha_k-\alpha_1)(\alpha_k-\alpha_2)\dots(\alpha_k-\alpha_{k-1})\,q_1(\alpha_k)$

as $\alpha_1 \neq \alpha_k, \alpha_2 \neq \alpha_k, \dots, \alpha_{k-1} \neq \alpha_k$

$\Rightarrow (\alpha_k-\alpha_1) \neq 0, (\alpha_k-\alpha_2) \neq 0, \dots, (\alpha_k-\alpha_{k-1}) \neq 0$.

$\Rightarrow q_1(\alpha_k) = 0$

$\Rightarrow \alpha_k$ is a root of $q_1(x)$.

$\Rightarrow q_1(x) = (x - \alpha_k) h(x)$ for some $h(x)$.

So from (3) Put $q_1(x)$ in eq. (2)

$f(x) = (x-\alpha_1)(x-\alpha_2)\dots\dots(x-\alpha_{k-1})(x-\alpha_k)h(x)$

$\Rightarrow (x-\alpha_1)(x-\alpha_2)\dots\dots(x-\alpha_{k-1})(x-\alpha_k) \mid f(x)$

Hence proved.

MSE

Theorem: Let $f(x)$ & $g(x)$ be two polynomials. Then α is the common root of $f(x)$ and $g(x)$ iff α is root of gcd of $f(x)$ & $g(x)$.

Example for understanding:-
$f(x) = x^2 - 1$, $g(x) = x - 1$
$= (x+1)(x-1)$

$h(x) = gcd[f(x), g(x)] = x - 1$
1 is the only root of $h(x)$

$\Rightarrow$ 1 is the only common root of $f(x)$ & $g(x)$.

Proof: Suppose α is the common root of $f(x)$ & $g(x)$.

$\Rightarrow$ $x - \alpha | f(x)$ [by Previous thm.]

and $x - \alpha | g(x)$ [by Previous theorem].

$\Rightarrow$ $x - \alpha$ is a common divisor of $g(x)$ & $f(x)$.
and let $h(x) = gcd[f(x), g(x)]$.
then,
$x - \alpha | h(x)$. [by def. of gcd].

$\Rightarrow$ α is a root of $h(x)$.

Converse:- Let $h(x) = gcd[f(x), g(x)]$. &
α is root of $h(x)$.
T.P. α is a common root of $f(x)$ & $g(x)$.

as α is a root of $h(x) \Rightarrow x-\alpha \mid h(x)$ — ①
as $h(x) = \gcd[f(x), g(x)]$.
$\Rightarrow h(x) \mid f(x)$ ↳② and $h(x) \mid g(x)$ ↳③

from ① & ②.
$x-\alpha \mid f(x)$.

Example
$2 \mid 4$ and $4 \mid 8$.
$\Rightarrow 2 \mid 8$

from ① & ③

$x-\alpha \mid g(x)$.

$\Rightarrow$ α is a root of $f(x)$ & $g(x)$.
$\Rightarrow$ α is a common root of $f(x)$ & $g(x)$.

* Multiple Roots:-

let $f(x)=0$ be a given polynomial equation. Then α is called a multiple root of $f(x)$ with multiplicity 'm'. of $f(x)=0$ iff.

$$(x-\alpha)^m \mid f(x) \quad \text{but} \quad (x-\alpha)^{m+1} \nmid f(x)$$

$(x-\alpha)^{m+1}$ does not divide $f(x)$.

Example:- $f(x) = (x-1)^3 (x-2)^2 (x-5)$

$\alpha = 2$

$$(x-2)^2 \mid f(x) \quad \text{but} \quad (x-2)^3 \nmid f(x).$$

$\Rightarrow \alpha = 2$ is a multiple root of $f(x)$ with multiplicity '2'.

* If $m=1$ for some root α_1, then α_1 is called simple root.

* If $m=2$ for some root α_2, then α_2 is called double root.

* If $m=3$ for some root α_3, then α_3 is called triple root.

Theorem:- α is a root of multiplicity m of a polynomial equation $f(x)=0$ iff their exist a polynomial $g(x)$ such that

$$f(x) = (x-\alpha)^m g(x) \; ; \; g(\alpha) \neq 0$$

Proof:- Let α be a root of multiplicity m of polynomial eq. $f(x)=0$.
then by def.

$$(x-\alpha)^m / f(x) \quad \text{but} \quad (x-\alpha)^{m+1} \nmid f(x) \; - (*)$$

$\Rightarrow f(x) = (x-\alpha)^m \cdot g(x). \; - (1)$

If possible assume that $g(\alpha)=0$

$\Rightarrow \alpha$ is a root of $g(x)$.

$\Rightarrow x-\alpha / g(x)$.

$\Rightarrow g(x) = (x-\alpha) h(x)$.

Put value of $g(x)$ in eq. (1)

$f(x) = (x-\alpha)^m (x-\alpha) h(x)$

DATE
PAGE

$f(x) = (x-\alpha)^{m+1} h(x).$

$\Rightarrow (x-\alpha)^{m+1} / f(x).$

Which is contradiction to (*).
hence $g(\alpha)$ can not be zero.

$\Rightarrow f(x) = (x-\alpha)^m g(x)$; $g(\alpha) \neq 0.$

Converse: Given $f(x) = (x-\alpha)^m g(x)$; $g(\alpha) \neq 0$

T.P α is a root of $f(x)$ of multiplicity 'm'.

i.e. $(x-\alpha)^m / f(x)$ but $(x-\alpha)^{m+1} \nmid f(x)$

from Given statement, it is clear that $(x-\alpha)^m / f(x)$

If possible assume that $(x-\alpha)^{m+1} / f(x).$

$\Rightarrow f(x) = (x-\alpha)^{m+1} h(x),$ — (2)

also $f(x) = (x-\alpha)^m g(x).$ — (3) [given].

from (2) and (3)

$(x-\alpha)^{m+1} h(x) = (x-\alpha)^m g(x).$

$(x-\alpha) h(x) = g(x)$

$\Rightarrow x-\alpha / g(x).$

$\Rightarrow$ α is root of $g(x)$ $\Rightarrow$ $g(\alpha) = 0$
Which is a contradiction to given statement.

Hence $(x-\alpha)^{m+1} \nmid f(x)$.

hence α is a root of $f(x)$ of multiplicity m.

2/Feb/2022.

Theorem: If $a_1, a_2, \ldots, a_n$ are distinct roots of $f(x)$ with multiplicity $m_1, m_2, \ldots, m_n$, then

$$(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_n)^{m_n} \mid f(x).$$

For Understanding:

Example: Roots of are 1, 2 & 5.

1 is a root of $f(x)$ with multiplicity 2.

2 " " " " " " 4.

5 " " " " " " 3.

So, we can say that

$$(x-1)^2(x-2)^4(x-5)^3 \mid f(x).$$

$$\Rightarrow f(x) = (x-1)^2(x-2)^4(x-5)^3 g(x).$$

Proof: We Prove the theorem by PMI on n.

Case 1: When $n=1$.

i.e. a_1 is the root of $f(x)$ with multiplicity m_1.

$\Rightarrow$ by def $(x-a_1)^{m_1} \mid f(x)$ but $(x-a_1)^{m_1+1} \nmid f(x)$.

So, Result is true for $n=1$.

DATE
PAGE

Case 2:- Assume that the Result is true for $n=k$.
i.e. if $a_1, a_2, a_3, \ldots, a_k$ are roots of $f(x)$ with multiplicity $m_1, m_2, m_3, \ldots, m_k$, then

$$(x-a_1)^{m_1}(x-a_2)^{m_2}(x-a_3)^{m_3}\ldots(x-a_k)^{m_k} \mid f(x).$$

$$\Rightarrow f(x) = (x-a_1)^{m_1}(x-a_2)^{m_2}(x-a_3)^{m_3}\ldots(x-a_k)^{m_k}\, g(x) \quad \text{— (1)}$$

Case 3:- We will have the result for $n=k+1$.

i.e. let $a_1, a_2, a_3, \ldots, a_{k+1}$ be roots of $f(x)$ with multiplicity $m_1, m_2, m_3, \ldots, m_{k+1}$.

T.P $(x-a_1)^{m_1}(x-a_2)^{m_2}(x-a_3)^{m_3}\ldots(x-a_k)^{m_k}(x-a_{k+1})^{m_{k+1}} \mid f(x)$

i.e. $f(x) = (x-a_1)^{m_1}(x-a_2)^{m_2}(x-a_3)^{m_3}\ldots(x-a_k)^{m_k}(x-a_{k+1})^{m_{k+1}}\, h(x)$.

$\Rightarrow$ Put $x = a_{k+1}$ in eq. (1)

$$f(a_{k+1}) = (a_{k+1}-a_1)^{m_1}(a_{k+1}-a_2)^{m_2}\ldots(a_{k+1}-a_k)^{m_k}\, g(a_{k+1}).$$

$$0 = (a_{k+1}-a_1)^{m_1}(a_{k+1}-a_2)^{m_2}\ldots(a_{k+1}-a_k)^{m_k}\, g(a_{k+1}).$$

$f(a_{k+1}) = 0$ because we assume it above a_{k+1} is a root of $f(x)$.

but $a_{k+1} \neq a_1 \Rightarrow a_{k+1} - a_1 \neq 0$

$a_{k+1} \neq a_2 \Rightarrow a_{k+1} - a_2 \neq 0$

$\vdots$

$a_{k+1} \neq a_k \Rightarrow a_{k+1} - a_k \neq 0$.

hence $g(a_{k+1}) = 0$

$\Rightarrow a_{k+1}$ is a root of $g(x)$.

as a_{k+1} is a root of $f(x)$ with multiplicity m_{k+1}.
$\Rightarrow$ a_{k+1} is a root of $g(x)$ " " m_{k+1}.

$\Rightarrow$ by def. $(x-a_{k+1})^{m_{k+1}} / g(x)$.

$\Rightarrow g(x) = (x-a_{k+1})^{m_{k+1}} h(x)$.

Put $g(x)$ in eq. ①

$$f(x) = (x-a_1)^{m_1}(x-a_2)^{m_2}(x-a_3)^{m_3} \dots (x-a_k)^{m_k}(x-a_{k+1})^{m_{k+1}} h(x)$$

$$\Rightarrow (x-a_1)^{m_1}(x-a_2)^{m_2}(x-a_3)^{m_3} \dots (x-a_k)^{m_k}(x-a_{k+1})^{m_{k+1}} / f(x).$$

Hence Result is true for $n = k+1$.
$\Rightarrow$ By PMI, the result is true for all n.

DATE
PAGE

9/Feb/2022. MSc

Theorem:-

If 'a' is a repeated root of $f(x)=0$, then it is ~~the~~ a root of $f(x)$ and $f'(x)$ and vice-versa.

Proof:-

Let a be repeated root of $f(x)$ with multiplicity 'm' (clearly $m>1$).

$\Rightarrow (x-a)^m / f(x)$.

$\Rightarrow f(x) = (x-a)^m g(x)$

$f'(x) = m(x-a)^{m-1} g(x) + (x-a)^m g'(x)$.

$f'(a) = m(a-a)^{m-1} g(a) + (a-a)^m g'(a)$.

$f'(a) = 0$

$\Rightarrow a$ is a root of $f'(x)$.

<u>Converse</u>:- Let a be root of both $f(x)$ & $f'(x)$.

$\Rightarrow f(x) = (x-a)g(x)$ — ①

$f'(x) = g(x) + (x-a)g'(x)$

given $f'(a) = 0$

Put $x = a$.

$\Rightarrow f'(a) = g(a) + (a-a)g'(a)$.

$0 = g(a)$.

$\Rightarrow a$ is a root of $g(x)$.

$\Rightarrow g(x) = (x-a)h(x)$.

Put $g(x)$ in eq. ①

$f(x) = (x-a)^2 / R(x)$.

MSQ.

Theorem:
'a' is a repeated root of Polynomial eq^n $f(x)=0$ iff 'a' is a root of gcd of $f(x)$ and $f'(x)$.

Proof: Let 'a' be repeated of $f(x)$.
By Previous theorem, a is root of $f(x)$ as well as $f'(x)$

$\Rightarrow f(x) = (x-a)g(x)$.

$f'(x) = (x-a)h(x)$

$\Rightarrow (x-a)/f(x), \quad (x-a)/f'(x)$

$\Rightarrow (x-a)$ is common divisor of $f(x)$ & $f'(x)$.

So by def. $(x-a) | gcd[f(x), f'(x)]$.

$\Rightarrow (x-a)/d(x)$. $\quad [gcd(f(x), f'(x)] = d(x)]$.

$\Rightarrow d(x) = (x-a)b(x)$.

$\Rightarrow$ a is a root of $d(x)$.

Converse: Let $d(x) = gcd[f(x), f'(x)]$.

Let 'a' be root of $d(x)$.

$\Rightarrow (x-a) | d(x)$

Also $d(x)/f(x)$. & $d(x)/f'(x)$.

by transitivity.

$(x-a)/f(x)$ & $(x-a)/f'(x)$.

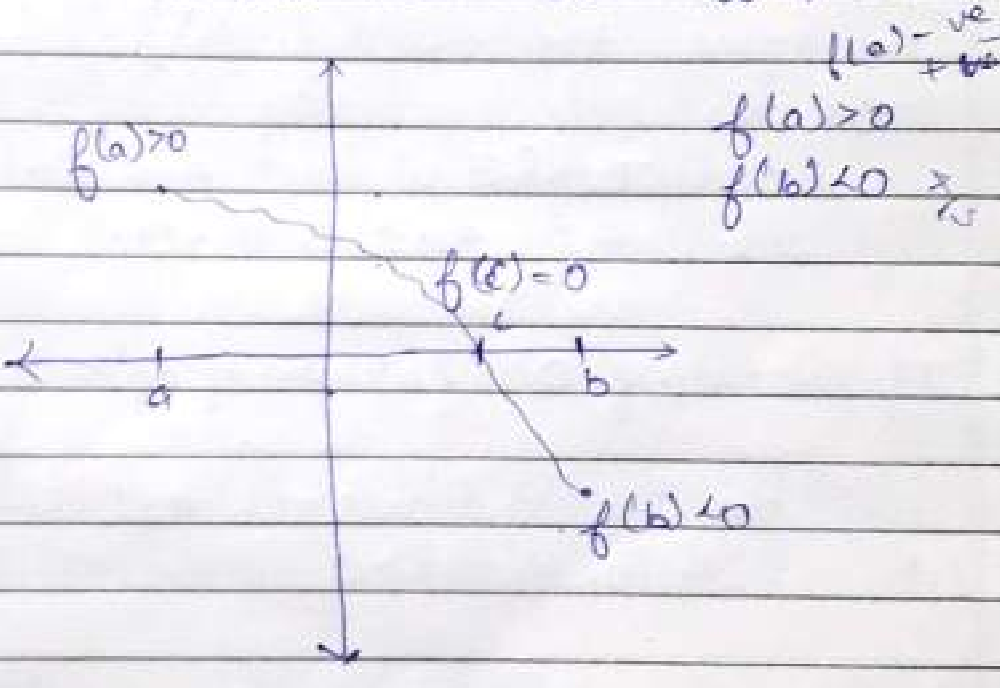
DATE
PAGE

$\Rightarrow$ a is a root of $f(x)$ & $f'(x)$.
So by previous theorem, a is repeated root of $f(x)$.

11/02/2022

* Intermediate Value Theorem :-

If two real quantities a & b be substituted in a polynomial $f(x)$ for the unknown variable x and if they furnish results having different signs one plus and the other minus, then the eqn. $f(x)=0$ must have atleast one real root between a & b.



Corallary :- If there exist no real quantity which substituted for x, make $f(x)=0$, then $f(x)$ must be +ve for every real value of x.
(Assuming that leading coff. is +ve).

Scanned with CamScanner

Theorem: Every Equation of odd degree has atleast one real root of a sign opposite to that of its last term.

Proof: $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$

Where n is odd.

$x = -\infty$, $f(x) \to -\infty$

$x = 0$, $f(x) = a_n$

$x = \infty$, $f(x) \to \infty$

If a_n is +ve $a_n > 0$, $f(-\infty)$ & $f(0)$ are of opposite sign.

$\Rightarrow$ by I.V.P $\exists$ at least one real root α of $f(x)$ s.t. $\alpha \in (-\infty, 0) \Rightarrow \alpha < 0$

If $a_n < 0$, $f(0)$ & $f(\infty)$ are of opposite sign.

$\Rightarrow$ by I.V.P $\exists$ at least one real root β of $f(x)$ s.t $\beta \in (0, \infty) \Rightarrow \beta > 0$

Theorem:- Every Equation of even degree with last term negative has atleast 2 real roots, one +ve & the other negative.

Proof: Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$

Where $a_n < 0$ & n is even.

$x = -\infty$, $f(x) \to \infty$

$x = 0$, $f(x) = a_n < 0$

$x = \infty$, $f(x) \to \infty$

as $f(\infty) > 0$, $f(0) < 0$, so by I.V.P., $\exists$ atleast one real root α of $f(x)$ s.t. $\alpha \in (-\infty, 0) \Rightarrow \alpha < 0$.

as $f(0) < 0$, $f(\infty) > 0$, so by I.V.P. $\exists$ atleast one real root β of $f(x)$ s.t. $\beta \in (0, \infty) \Rightarrow \beta > 0$.

FUNDAMENTAL THEOREM OF ALGEBRA :-

If $f(x)$ is a non-constant polynomial of degree n, then no. of real roots of $f(x)$ is atmost n.

(For Real Case).

If $f(x)$ is a non-constant polynomial of degree n, then it has exactly n roots.

(For Complex Case).

MSC Result :- In a real polynimal, imaginary roots occurs in conjugate pairs.

Proof :- Let $f(x)$ be a real polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n;$$

Where all a_i's are real.

Let $x = \alpha + i\beta$ be a root of $f(x)$. — $(*)$

Let $\phi(x) = [x - (\alpha + i\beta)][x - (\alpha - i\beta)]$.

$$= (x - \alpha - i\beta)(x - \alpha + i\beta)$$

$$= (x - \alpha)^2 - (i\beta)^2$$

$$= (x - \alpha)^2 + \beta^2$$

by Division Algorithm on $f(x)$ & $\phi(x)$.

$f(x) = \phi(x)q(x) + r(x)$; $\deg r(x) < 2$ $\to$ (1)

$\Rightarrow r(x) = r_1 x + r_2$; where r_1 & r_2 are Real val. constant. $\to$ (2)

Put $x = \alpha + i\beta$ in (1)

$f(\alpha+i\beta) = \phi(\alpha+i\beta)q(\alpha+i\beta) + r(\alpha+i\beta)$.

$0 = 0 + r(\alpha+i\beta)$.

$0 = r_1(\alpha+i\beta) + r_2$. [using eq. (2)]

$\Rightarrow (r_1\alpha + r_2) + i\beta r_1 = 0 + 0i$

$\Rightarrow \beta r_1 = 0$ & $r_1\alpha + r_2 = 0$

$\Rightarrow r_1 = 0$ $\quad$ $r_2 = 0$.

($\because \beta$ is not zero)

So, eq. (2)

$r(x) = 0$

Put $r(x) = 0$ in eq. (1)

$f(x) = \phi(x)q(x)$.

$\Rightarrow \phi(x) | f(x)$.

$\Rightarrow [x - (\alpha+i\beta)][x - (\alpha - i\beta)] \,/\, f(x)$.

Scanned with CamScanner

Continuation Or Variation of signs :-

Let $f(x)$ be a polynomial Whose terms are arranged in descending powers of x. Then if a +ve sign is followed by a +ve sign and a -ve sign is followed by a -ve sign, the Equation is said to ~~be~~ have Continuation or permanence in sign.

If a +ve sign is followed by a -ve sign or a -ve sign is followed by a +ve sign, then the equation is said to ~~be~~ have variance or change in sign.

Example :- $f(x) = x^5 - 3x^2 + 5x^3 - 3x + 2x^4 - 1$

$$= x^5 + 2x^4 + 5x^3 - 3x^2 - 3x - 1$$

Continuous = 4
Variance = 1.

$f(x) = -2x^4 + x^3 - 4x^2 + 10x - 3.$

Variance = 4.

$f(x) = -2x^4 + x^3 + 0x^2 + 0x - 3$

Continuous = 2
Variance = 2

Result :- In a Complete Polynomials. No. of variations plus No. of Continuation equals to degree of polynomial.

2) No. of variation + No. of Continuation = Degree of Polynomial.

Example:- $f(x) = 2x^4 - x^3 - 2x^2 + 10x - 3x^0$

Variance = 3

Continuous = 1.

let

$f(-x) = 2x^4 + x^3 - 2x^2 - 10x - 3.$

Continuous = 3

Variance = 1.

Result:* No. of Continuations in $f(x)$ is same as no. of variations in $f(-x)$. & no. of variations in $f(x)$ is same as no. of continuous in $f(-x)$.

Result:- If between any 2 like signs, any no. of +ve or -ve signs are introduced, then no. of variations in the result will be a +ve even no.

Example:-

If between any 2 unlike signs, any no. of +ve or -ve signs are introduced, then the no. of variations in the result will be a +ve odd no.

Examples:-

DATE
PAGE

Eg: $f(x) = 4x^5 - 3x^3 + 2x^2 - 2x + 8.$

Cont. $= 0$

Variance $= 4$

Let $\alpha > 0$,

$$(x-\alpha) f(x) = 4x^6 - 3x^4 + 2x^3 - 2x^2 + 8x - 4\alpha x^5 + 3\alpha x^3 - 2\alpha x^2 + 2\alpha x - 8\alpha$$

$$= 4x^6 - 4\alpha x^5 - 3x^4 + (2+3\alpha)x^3 + (-2-2\alpha)x^2 + (2\alpha+8)x - 8$$

$$= 4x^6 - 4\alpha x^5 - 3x^4 + (2+3\alpha)x^3 - (2+2\alpha)x^2 + (2\alpha+8)x - 8$$

Variance $= 5$.

Con. $= 1$.

(2) $f(x) = x^2 + x + 1$

$$(x-1) f(x) = x^3 + x^2 + x - x^2 - x - 1.$$

$$= x^3 - 1$$

Variance $= 1$.

(3) $f(x) = 4x^5 + 3x^4 + 2x^3 + 6x^2 + 3x + 9.$

$$(x-\alpha) f(x) = 4x^6 + 3x^5 + 2x^4 + 6x^3 + 3x^2 + 9x - 4\alpha x^5 - 3\alpha x^4 - 2\alpha x^3 - 6\alpha x^2 - 3\alpha x - 9\alpha$$

$$= 4x^6 + (3-4\alpha)x^5 + (2-3\alpha)x^4 + (6-2\alpha)x^3 + (3-6\alpha)x^2 + (9-3\alpha)x - 9\alpha$$

~~Variation~~ $= + \quad \pm \quad \pm \quad \pm \quad \pm \quad \pm \quad -$

~~Continuation~~ $= 5$

So, No. of Variations = odd.

DATE
PAGE

Result:- If a polynomial $f(x)$ is multiplied by a factor $x-\alpha$ where $\alpha>0$; the no. of variations in the resulting polynomial shall ~~be~~ exceed those in $f(x)$ by atleast 1 or always by an odd number.

Descarte's Rule Of Signs:-

Statement:- The Number of +ve real roots of $f(x)=0$, cannot exceed the number of variations in sign of $f(x)$ & is less than the number of variations by an even no.

Example:- $f(x) = x^2+2x+1$. [No. of variation = 0]

no. of +ve real root $\leq$ no. of variations.

$f(x) = x^2-2x+1$.

no. of +ve real root $\leq$ no. of variations.

no. of variations = 2.

* If no. of variations is 4 then +ve Real root will be 4, 2, 0.

Observation:-

$f(x) = x^2-2x+1$

Roots of $f(x)$ is 1, 1.

$f(-x) = x^2+2x+1$

Roots of $f(-x)$ is $-1, -1$.

No. of +ve real root of $f(-x)$ = No. of -ve real roots of $f(x)$.

DATE
PAGE

The No. of Negative real roots of a polynomial $f(x)$ same as the no. of +ve real roots of $f(-x)$.

$\because$ if roots of $f(x)$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$.
then roots of $f(-x)$ are $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$.

Example:- Prove that $f(x) = x^3+3x+2$ has 2 non-real roots.

S:- No. of Variations in $f(x) = 0$
$\Rightarrow$ no +ve real roots.

$f(-x) = -x^3 - 3x + 2$.
No. of variations in $f(-x) = 1$.
$\Rightarrow$ No. of +ve real root of $f(-x)$.
= No. of -ve real root of $f(x) = 1$.

14/02/22.

Proof of Descarte's Rule of Signs:-
Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be +ve reals roots of $f(x)$.
$(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_m)$ be product of the factors.
Let $\phi(x)$ be product of factors of $f(x)$, corresponding to -ve & complex roots.
$\Rightarrow f(x) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_m)\phi(x)$.

So $\phi(x)$ has factors of the form $x+h$ & $x^2-2\alpha x+(\alpha^2+\beta^2)$.
Where $h>0$. So the ~~coff~~ leading coefficient of $\phi(x)$ and the Constant term are always +ve.
$\Rightarrow$ No. of variation of signs in $\phi(x)$ is +ve even no. say $2r$
$\Rightarrow$ No. of variations of signs in $(x-\alpha_1)\phi(x)$ is increased by an odd no. say $2r_1+1$.

The No. of variations of signs in $(x-\alpha_2)(x-\alpha_1)\phi(x)$ is increased by an odd no. say $2r_2+1$.

and so on.
Hence no. of variations of signs in $f(x)$.

$$= 2r + (2r_1+1) + (2r_2+1) + \cdots + (2r_m+1).$$

where $r, r_1, r_2, r_3 \ldots r_m \geq 0$

$$= 2(r + r_1 + r_2 + \cdots + r_m) + m.$$

$\Rightarrow$ No. of variations of signs in $f(x) \geq m$.
(m = no. of +ve real roots)

No. of variations of signs in $f(x)$ - no. of +ve real roots of f

$$= 2(r + r_1 + \cdots + r_m) + m - m.$$
$$= 2(r + r_1 + \cdots + r_m).$$
= even no.

Example:-

① Find the nature of roots of

$$f(x) = x^5 - 3x^4 + 4x^3 + 2x^2 + x - 8$$

No. of +ve real roots ≤ 3

$$f(-x) = -x^5 - 3x^4 - 4x^3 + 2x^2 - x - 8$$

No. of +ve real roots of $f(-x)$ = no. of -ve real roots of
No. of -ve real roots of $f(x) \leq 2$.

DATE
PAGE

Possible Roots	+ve	-ve	complex roots	Total = degree of variable
	3	2	0	5
	3	0	2	5
	1	2	2	5
	1	0	4	5

① If a Polynomial $f(x)$ has all terms +ve, then it cannot have any +ve root.

② If a Polynomial $f(x)$ has terms with alternative signs

(x). $$f(x) = \begin{cases} -a_0x^n + a_1x^{n-1} - \cdots + a_{n-1}x - a_n & \text{if } n \text{ is even.} \\ -a_0x^n + a_1x^{n-1} - \cdots - a_{n-1} + a_n & \text{if } n \text{ is odd.} \end{cases}$$

$$f(-x) = \begin{cases} -a_0x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n & \text{if } n \text{ is even.} \\ a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n. & \text{if } n \text{ is odd} \end{cases}$$

Example :-

n is even, $f(x) = -a_1x^4 + a_2x^3 - a_3x^2 + a_4x - a_5$

$$f(-x) = -a_1x^4 - a_2x^3 - a_3x^2 - a_4x - a_5.$$

n is odd, $f(x) = -a_1x^5 + a_2x^4 - a_3x^3 + a_4x^2 - a_5x + a_6$

$f(-x) = a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6.$

no. of variations of signs in $f(-x) = 0$
$\Rightarrow$ no. of neg. real roots of $f(x) = 0$

(3) If an Equation involves only even powers of x then $f(x) = f(-x)$.
$\Rightarrow$ No. of variations of signs in $f(x)$ = No. of variation of signs in $f(-x)$

(4) If a polynomial is complete then the no. of continuation in $f(x)$ is same as no. of variations of signs in $f(-x)$.

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$

Result:- If two numbers a & b substituted for a polynomial $f(x)$ give results with opposite signs, then an odd number of real roots of $f(x) = 0$ lies between a & b.

Proof:- Let $f(x) = 0$ has m^{real} roots between a & b say $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_m$.
Then $f(x) = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_m)\cdot\phi(x)$ — (1)

$f(a) = (a-\alpha_1)(a-\alpha_2)\dots(a-\alpha_m)\phi(a)$
$f(b) = (b-\alpha_1)(b-\alpha_2)\dots(b-\alpha_m)\phi(b)$

If $\phi(a)$ & $\phi(b)$ are of opposite signs, by I.V.P, there is a root say α of $\phi(x)$ between a & b.
i.e. $\alpha \in (a,b)$ and $\phi(\alpha) = 0$.
$\Rightarrow f(\alpha) = 0 \Rightarrow \alpha$ is root of $f(x)$.

DATE
PAGE

$\Rightarrow f$ has $m+1$ roots in a and b.
$\therefore$ This is contradiction.
$\Rightarrow \phi(a)$ & $\phi(b)$ are of same sign ✓

$\Rightarrow (a-\alpha_1)(a-\alpha_2)\ldots(a-\alpha_m)$ & $(b-\alpha_1)(b-\alpha_2)\ldots(b-\alpha_m)$ are of opposite sign.

also $a < \alpha_i < b$ $\quad a > \alpha_i \rightarrow$
$\forall\ 1 \le i \le m.$ $\quad b > \alpha_i$

$\Rightarrow a - \alpha_i < 0$
$\forall\ 1 \le i \le m$

$a - \alpha_i > 0$ +ve $\rightarrow$ +ve
$b - \alpha_i > 0$ +ve
$a < \alpha_i < b$
$\forall\ 1 \le i \le m$

$\Rightarrow b - \alpha_i > 0\ \forall$
$1 \le i \le m.$

$\alpha_i < a$ ✓
$\alpha_i < b$

$\Rightarrow \prod_{i=1}^{m} (b-\alpha_i) > 0$ ✓ +ve

$0 < b - \alpha_1$

$\rightarrow$ odd.

$\Rightarrow \prod_{i=1}^{m} (a-\alpha_i) < 0$ ✓ -ve

$(a-\alpha_i)$

as each $(a-\alpha_i) < 0$
$\Rightarrow m$ is odd.

Result: If two numbers a & b substituted for a polynomial $f(x)$ give results with same signs, then an even number of real roots of $f(x) = 0$ lies between a & b.

DATE
PAGE

* Relation between Roots & Coefficients of a Polynomial

let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n;\ a_0 \neq 0$

$ax^2+bx+c=0$, α, β roots

$\alpha+\beta = \frac{-b}{a}$

$\alpha\beta = \frac{c}{a}$

let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be roots of $f(x)$.

$a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) = f(x)$ — ①

$a_0\left[x^n + (-1)^1(\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n)x^{n-1} + \sum_{i\neq j}\alpha_i\alpha_j x^{n-2} - \sum_{\substack{i,j,k \\ \text{different}}}\alpha_i\alpha_j\alpha_k x^{n-3} + \cdots + (-1)^n\alpha_1\alpha_2\alpha_3\cdots\alpha_n x^0\right] = f(x).$

$2x^2+5x+3$
$=(2x+3)(x+1)$
$x = -\frac{3}{2}, -1$

$\left(x+\frac{3}{2}\right)(x+1)$

$x^2+\frac{3}{2}x+x+\frac{3}{2}$

$x^2+\frac{5}{2}x+\frac{3}{2}$

$2\left(x^2+\frac{5}{2}+\frac{3}{2}\right)$

$\Rightarrow x^n + (-1)^1(\alpha_1+\alpha_2+\cdots+\alpha_n)x^{n-1} + (-1)^2\sum_{i\neq j}\alpha_i\alpha_j + (-1)^3\sum_{i,j,k}\alpha_i\alpha_j\alpha_k x^{n-3} \cdots + (-1)^n\alpha_1\cdots\alpha_n x^0 = \frac{f(x)}{a_0}$

$(x-\alpha_1)\underline{(x-\alpha_2)(x-\alpha_3)}$

$= (x^2-(\alpha_2+\alpha_3)x+\alpha_2\alpha_3)$

$= x^3 - \alpha_1x^2 - (\alpha_2+\alpha_3)x^2$

Coff. x^2

using eq. ①

$\Rightarrow x^n + (-1)(\alpha_1+\alpha_2+\cdots+\alpha_n)x^{n-1} + (-1)^2\sum_{i\neq j}\alpha_i\alpha_j + (-1)^3\sum_{i,j,k}\alpha_i\alpha_j\alpha_k x^{n-3} \cdots + (-1)^n\alpha_1\cdots\alpha_n x^0 = x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0}.$

$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$

Coff of x^2

Comparing coeff. of same powers of x on both side

$\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n = -\frac{a_1}{a_0}$

$-\frac{2nd}{1st}$

$\sum_{i\neq j}\alpha_i\alpha_j = \frac{a_2}{a_0}$

~~coeff~~ coeff $= +\frac{3rd}{1st}$

$\sum_{i,j,k}\alpha_i\alpha_j\alpha_k = \frac{-a_3}{a_0}$

$-\frac{4th}{1}$

Scanned with CamScanner

$\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$

Eg:- $4x^3 + 3x^2 - 5x + 3 = 0$

$\alpha_1 + \alpha_2 + \alpha_3 = -3/4$

$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 = -5/4$

$\alpha_1 \alpha_2 \alpha_3 = \frac{-3}{4}$

[odd] that's why start from -ve

Eg:- $3x^4 + 4x^3 + 3x^2 - 5x + 3 = 0$

$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -\frac{4}{3}$

$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = \frac{3}{3} = 1.$

$\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2\alpha_3 + \quad = \frac{5}{3}$

$\alpha_1\alpha_2\alpha_3\alpha_4 = \frac{3}{3} = 1.$

Problems

DATE
PAGE

① Show that the equation $x^{11} - x^6 + 4x^5 - 5 = 0$ has atleast 8 non-real roots can a real root of the equation be negative? Justify.

Sol:- $f(x) = x^{11} - x^6 + 4x^5 - 5$

Variations in $f(x) = 3$

$\therefore$ no. of +ve real roots of $f(x) \leq 3$

$f(-x) = -x^{11} - x^6 - 4x^5 - 5$

variation in $f(-x) = 0$

$\therefore$ no. of -ve real roots of $f(x) \leq 0$

We know, Total roots of $f(x) = 11$

As no. of +ve real roots of $f(x) \leq 3$

no. of -ve real roots of $f(x) \leq 0$.

$\Rightarrow$ Remaining 8 roots are non-real roots.

No, there is no -ve real roots.

② Show that the equation $2x^7 + 3x^4 + 3x + k = 0$ has atleast 4 non-real roots for all values of k.

Sol:- If $k > 0$

then, $f(x) = 2x^7 + 3x^4 + 3x + k$

Variations in $f(x) = 0$

no. of +ve real roots of $f(x) \leq 0$

$f(-x) = -2x^7 + 3x^4 - 3 + k$

Variations in $f(-x) = 3$

no. of -ve real roots of $f(x) \leq 3$.

We know, Total roots of $f(x) = 7$.

As no. of +ve real roots of $f(x) \leq 0$

no. of -ve real roots of $f(x) \leq 3$

$\Rightarrow$ Remaining 4 roots are non-real roots.

If $k < 0$

then, $f(x) = 2x^7 + 3x^4 + 3x - k$

Variations in $f(x) = 1$.

$\therefore$ no. of +ve real roots of $f(x) \leq 1$.

$f(-x) = -2x^7 + 3x^4 - 3x - k$

Variations in $f(-x) = 2$

$\therefore$ no. of -ve real roots of $f(x) \leq 2$.

Total roots of $f(x) = 11$

as no. of +ve real roots of $f(x) \leq 1$

no. of -ve real roots of $f(x) \leq 2$.

$\Rightarrow$ Remaining 4 roots are non-real roots.

If $k = 0$

then, $f(x) = 2x^7 + 3x^4 + 3x$

Variations in $f(x) = 0$

$\therefore$ no. of +ve real roots $= 0$

then, $f(-x) = -2x^7 + 3x^4 - 3$

Variations in $f(x) = 2$

$\therefore$ no. of -ve real roots of $f(x) \leq 2$

$f(0) = 0$.

0 is also a root of $f(x)$.

Total Roots $= 7$.

no. of +ve real roots of $f(x) = 0$

no. of -ve real roots of $f(x) \leq 2$

One root $= 0$.

$\therefore$ Remaining 4 roots are non-real roots.

(3) Show that the equation, $x^n - a = 0$; $a > 0$ has

i) exactly two real roots if n is even.

ii) exactly ~~two~~ one real roots if n is odd.

:- $f(x) = x^n - a$

(i) If n is even

$f(x) = x^n - a$

Variations in $f(x) = 1$

$\therefore$ no. of +ve real roots of $f(x) \leq 1$

$f(-x) = (-x)^n - a$

$= x^n - a$

Variations in $f(-x) = 1$

$\therefore$ no. of -ve real roots of $f(x) \leq 1$.

Total root $= n$.

no. of +ve real roots of $f(x) \leq 1$

no. of -ve real roots of $f(x) \leq 1$

Total real roots $= 2$.

(ii) If n is odd.

$f(x) = x^n - a$

variation in $f(x) = 1$.

$\therefore$ no. of +ve real root of $f(x) \leq 1$

$f(-x) = (-x)^n - a$

$= -x^n - a$

Variation in $f(x) = 0$

$\therefore$ no. of -ve real roots of $f(x) = 0$

Total roots $= n$

no. of +ve real roots $= 1$

no. of -ve real roots $= 0$

Total real root $= 1$.

(4) Find the situation of roots of $x^3+x^2-2x-1=0$.

Sol:- $f(x) = x^3+x^2-2x-1$.

Variation of $f(x) = 1$.

no. of +ve real roots of $f(x) = 1$.

$f(-x) = -x^3+x^2+2x-1$

Variation of $f(-x) = 2$

no. of -ve real roots of $f(x) \leq 2$

For +ve real Roots:-

$f(0) = (0)^3+(0)^2-2(0)-1$

$= -1$

$f(0) = -1 < 0$.

$f(1) = 1+1-2-1$

$= -1 < 0$.

Root between $(0,1)$ is not lie.

$f(2) = 8+4-4-1$

$= 7 > 0$.

Root lies between $(1,2)$.

For -ve real roots

$f(-1) = -1+1+2-1$

$= 1 > 0$.

One -ve root lie b/w (

$f(-2) = -8+4+4-1$

$= -1 < 0$.

One -ve root lie b/w (-1,

Situation of roots is $(1,2), (-1,0), (-1,-2)$.

*(5) Find the real root's situation of $x^4-7x^2+18x-8$

Sol:- $f(x) = x^4-7x^2+18x-8 = 0$

Variations in $f(x) = 3$

$\therefore$ no. of +ve real roots of $f(x) \leq 3$.

DATE
PAGE

$f(-x) = x^4 - 7x^2 - 18x - 8.$

Variation of $f(x) = 1$

$\therefore$ no. of $-$ve real roots of $f(x) = 1.$

$f(0) = -8 < 0.$

For $+$ve real roots.

$f(1) = 1 - 7 + 18 - 8$

$= 4 > 0.$

one $+$ve roots of $f(x)$ in $(0,1)$.

$f(2) = 16 - 7(4) + 36 - 8$

$= 16 > 0$

$-1 \quad 0 \quad 1 \quad 2$

36

$f(3) = 81 - 63 + 54 - 8.$

$= 64 > 0$

for $-$ ve real roots.

$f(-1) = 1 - 7 - 18 - 8 < 0$

$f(-2) = 16 - 28 - 36 - 8 < 0.$

$f(-3) = 81 - 63 - 54 - 8 < 0.$

$f(-4) = 256 - 112 - 72 - 8 > 0.$

one $-$ve roots lie b/w $(-3, -4)$.

(6) Solve $x^6+2x^5-10x^4-12x^3+59x^2+10x-10=0$, if two of its roots are $i-3$ & $i+2$.

~~∴ we have roots, $i-3, i+3, i+2, i-2$ are four roots of equation.~~

~~Let α, β are also roots of $f(x)$.~~

~~$f(x) = x^6+2x^5-10x^4-12x^3+59x^2+10x-10$~~

~~Sum of roots $= -2$~~

~~$i-3+i+3+i+2+i-2+\alpha+\beta=0$.~~

~~$4i+\alpha+\beta = i$~~

we have roots,

$i-3, -i-3, i+2, -i+2$ are four root of equation.

$f(x) = x^6+2x^5-10x^4-12x^3+59x^2+10x-10$.

Let α & β are two other roots of $f(x)$.

⇒ Sum of roots $= -2$

$i-3-i-3+i+2-i+2+\alpha+\beta = -2$

$4-6+\alpha+\beta = -2$

$\boxed{\alpha+\beta=0}$ — (1)

⇒ Products of roots $= -10$.

$[(-3)^2-(i)^2][(2)^2-(i)^2]\,\alpha\beta = -10$

$(9+1)(4+1)\,\alpha\beta = -10$

$50\alpha\beta = -10$

$\boxed{\alpha\beta = -\frac{1}{5}}$ — (2)

from eq. (1)

$\alpha = -\beta$ — (3)

Put in eq. (2)

DATE
PAGE

$$(-\beta)(\beta) = -\frac{1}{5}$$

$$-\beta^2 = -\frac{1}{5}$$

$$\beta^2 = \frac{1}{5}$$

$$\boxed{\beta = \pm\frac{1}{\sqrt{5}}}$$

When $\beta = \frac{1}{\sqrt{5}}$ & $\beta = -\frac{1}{\sqrt{5}}$

from eq. ②	from eq. ②
$\alpha = -\frac{1}{\sqrt{5}}$	$\alpha = \frac{1}{\sqrt{5}}$.

So we have $\alpha = \pm\frac{1}{\sqrt{5}}$

$\beta = \pm\frac{1}{\sqrt{5}}$

Ques ⑦ Solve the equation $8x^3 - 12x^2 - 2x + 3 = 0$. Given that roots in A.P.

Sol:- $f(x) = 8x^3 - 12x^2 - 2x + 3$.

Let $a-d, a, a+d$ are roots of $f(x)$.

Sum of roots $= \frac{12}{8} = \frac{3}{2}$

$$a - d + a + a + d = \frac{3}{2}$$

$$3a = \frac{3}{2}$$

$$\boxed{a = \frac{1}{2}}$$

Products of roots $= -\frac{3}{8}$

$$a(a-d)(a+d) = -\frac{3}{8}$$

$$a(a^2 - d^2) = -\frac{3}{8}$$

$$\frac{1}{2}\left(\frac{1}{4} - d^2\right) = -\frac{3}{8}$$

$$\frac{1 - 4d^2}{4} = -\frac{6}{8} = -\frac{3}{4}$$

$$1 - 4d^2 = -3$$

$$-4d^2 = -4$$

$$d^2 = 1$$

$$\boxed{d = \pm 1}$$

Now, roots

when $d = 1$.

$-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ are roots of $f(x)$

When $d = -1$

$\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$ are roots of $f(x)$.

Ques:- Find the Value of k, given that $3x^3-27x^2+22x+k=0$ has roots in A.P.

let $a-d, a, a+d$ are the roots of $f(x)$

$f(x) = 3x^3 - 27x^2 + 22x + k$.

Sum of roots $= \frac{27}{3}$

$a-d+a+a+d = \frac{27}{3}$

$3a = \frac{27}{3}$

$\boxed{a=3}$

If a is 3, then its satisfy the equation.

$3x^3 - 27x^2 + 22x + k = 0$

$3(3)^3 - 27(3)^2 + 22(3) + k = 0$

$81 - 243 + 66 + k = 0$.

$-177 + 81 + k = 0$.

$-96 + k = 0$

$\boxed{k=96}$

Ques:- Find λ in $8x^3+\lambda x^2-18x+9=0$; given that one root negative of the other.

let $\alpha, -\alpha, \beta$ are roots.

$f(x) = 8x^3 + \lambda x^2 - 18x + 9$

Sum of roots $= \frac{-\lambda}{8}$

$\cancel{\alpha} \cancel{-\alpha} + \beta = \frac{-\lambda}{8}$

$\beta = \frac{-\lambda}{8}$

Scanned with CamScanner

DATE ______
PAGE ______

We assume β is a root of $f(x)$.

$\therefore f(\beta) = 8\beta^3 + \lambda\beta^2 - 18\beta + 9 = 0$

Put $\beta = \frac{-\lambda}{8}$

$= \frac{8(-\lambda)^3}{8 \cdot 8^2} + \frac{\lambda^3}{64} + \frac{18\lambda}{8} + 9 = 0$

$= \frac{-\lambda^3}{64} + \frac{\lambda^3}{64} + \frac{9\lambda}{4} + 9 = 0$

$= \frac{9\lambda}{4} + 9 = 0$

$\frac{9\lambda}{4} = -9$

$\boxed{\lambda = -4}$

$\beta = -\frac{1}{2}$

$\alpha^2\beta = \frac{-9}{8}$

$\alpha^2 = \frac{9}{4}$

$\alpha = \sqrt{\frac{9}{4}}$

$= \pm\frac{3}{2}$

Scanned with CamScanner

Solve :- $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$, given that roots are in A.P

Sol :- $f(x) = x^4 - 8x^3 + 14x^2 + 8x - 15$.

Let $a-3d, a-d, a+d, a+3d$ are roots of $f(x)$

4 no. in A.P

$a-3d, a-d, a+d, a+3d$

Sum of roots $= +8$

$a-3d+a-d+a+d+a+3d = +8$

$4a = +8$

$\boxed{a = +2}$

$(-1)^n \frac{a_n}{a_0}$

Product of roots $= -15$

$(a-3d)(a+3d)(a-d)(a+d) = -15$

$(a^2-9d^2)(a^2-d^2) = -15.$

$a^4 - a^2d^2 - 9d^2a^2 + 9d^4 = -15.$

$16 - 4d^2 - 36d^2 + 9d^4 = -15.$

$16 - 40d^2 + 9d^4 = -15$

$9d^4 - 40d^2 + 16 = -15$

$9d^4 - 40d^2 + 31 = 0.$

$16 + 15 = 31$

$d^2 = y$

$9y^2 - 40y + 31 = 0.$

$31 \times 9 = 279$

$279 = 3 \times 3 \times 31$

$9y^2 - 31y - 9y + 31 = 0$

$y(9y-31) - (9y-31) = 0$

$(y-1)(9y-31) = 0,$

$y = 1, \frac{31}{9}.$

$d^2 = 1$

$d = \pm 1$

$d^2 = \frac{31}{9}$

$d = \pm \frac{\sqrt{31}}{3}.$

DATE
PAGE

When $d = 1 \& -1$.

Ist root $a-3d = 2-3(1) = -1$

$d=-1$ $\quad a-3d = 2+3 = 5.$

$d=1$ 2nd root $a-d = 2-1 = 1$

$d=-1$ $\quad a-d = 2+1 = 3.$

$d=1$ 3rd root $a+d = 2+1 = 3$

$d=-1$ $\quad a+d = 2-1 = 1.$

$d=1$ 4th root $a+3d = 2+3 = 5$

$d=-1$ $\quad a+3d = 2-3 = -1.$

Q8- $x^3 - 2x^2 - 8x + 64 = 0$, given that it has three distinct roots of equal modulus.

S: Let $\alpha + i\beta, \alpha - i\beta, \gamma$ are roots of given eq.

Given

$|\alpha + i\beta| = |\alpha - i\beta| = |\gamma|$ $\qquad |\alpha+i\beta| = |\alpha - i\beta| = \sqrt{\alpha^2+\beta^2}$

$\sqrt{\alpha^2+\beta^2} = |\gamma|$

squaring

$\alpha^2+\beta^2 = \gamma^2$. — ①

Product of roots $= -64$.

$(\alpha - i\beta)(\alpha + i\beta)(\gamma) = -64$.

$(\alpha^2+\beta)^2(\gamma) = -64$

$(\gamma^2)(\gamma) = -64$ $\qquad$ [using eq. ①]

$\gamma^3 = -64$.

$\boxed{\gamma = -4}$

Sum of roots $= 2$.

$\alpha + i\beta + \alpha - i\beta + \gamma = 2$.

$2\alpha + \gamma = 2$

$2x - 4 = 2$

$x - 2 = 1$

$\boxed{x = 3}$

Put α & γ in eq. ①

$\alpha^2 + \beta^2 = \gamma^2$

$9 + \beta^2 = 16$

$\beta^2 = 16 - 9$

$\beta^2 = 7$

$\beta^2 = \pm\sqrt{7}.$

Ques:- Solve $2x^3 - 21x^2 + 42x - 16 = 0$, given that its roots in G.P.

:- $f(x) = 2x^3 - 21x^2 + 42x - 16.$

Let $\frac{a}{r}, a, ar$ are the roots of $f(x)$

Product of roots $= \frac{+16}{2} = +8.$

$a^3 = 8$

$\boxed{a = 2}$

Sum of roots $= \frac{21}{2}$

$\frac{a + ar + ar^2}{r} = \frac{21}{2}$

$2 + 2r + 2r^2 = \frac{21r}{2}.$

$4 + 4r + 4r^2 = 21r.$

$4r^2 - 17r + 4 = 0.$

$4r^2 - 16r - r + 4 = 0$

$4r(r-4) - (r-4) = 0$

$(4r-1)(r-4) = 0$

$\boxed{r = 4, \frac{1}{4}}$

When $r = 4$, $a = 2$

$\frac{1}{2}, 2, 8.$

When $r = \frac{1}{4}$, $a = 2$

$8, 2, \frac{1}{2}.$

$\frac{1}{2}, 2, 8$ are the roots of $f(x)$.

Ques: Construct a monic cubic polynomial such that $f(0) = 1$, $f(1) = 3$ and sum of roots is 2.

: Let $x^3 + ax^2 + bx + c = f(x)$

$f(0) = (0)^3 + a(0)^2 + b(0) + c = 1$

$\boxed{c = 1}$

$f(1) = 1 + a + b + c = 3.$

$= 1 + a + b + 1 = 3$

$a + b = 1. \quad - (1)$

$x^3 + ax^2 + bx + c = f(x)$

$\boxed{a = -2}$

$f(0) = 1$

$\boxed{c = 1}$

$a + b = 1$

Sum of roots $= -a$

$2 = -a$

$\boxed{a = -2}$

Put b in eq. (1)

$-2 + b = 1$

$\boxed{b = 3}$

Required Cubic equation:-

$f(x) = x^3 - 2x^2 + 3x + 1.$

Solve: $x^3 - 3x^2 - 3x + 1 = 0$, given that the reciprocal of roots are also the roots.

Let $\alpha, \frac{1}{\alpha}, \beta = \frac{1}{\beta}$ are the roots of equation

$\beta^2 = 1$

$\Rightarrow \beta = \pm 1$

Sum of roots $= 3$

$\alpha + \frac{1}{\alpha} + \beta = 3$

$\alpha + \frac{1}{\alpha} - 1 = 3$

$\frac{\alpha^2 + 1}{\alpha} = 4$

$\alpha^2 - 4\alpha + 1 = 0$

$\alpha = \frac{4 \pm \sqrt{16-4}}{2}$

$= \frac{4 \pm \sqrt{12}}{2}$

$\boxed{\alpha = 2 \pm \sqrt{3}}$

Product of roots $= -1$

$\alpha \cdot \frac{1}{\alpha} \cdot \beta = -1$

$\boxed{\beta = -1}$

Solve: $x^4 + 6x^3 - 16x^2 + 24x - 80 = 0$ given that two of its roots are purely imaginary.

$\therefore$ Let $\alpha, \beta, \gamma i, -\gamma i$ are roots of equation

Sum of roots $= -6$

$\alpha + \beta + \gamma i - \gamma i = -6$

$\alpha + \beta = -6$ — (1)

DATE
PAGE

Let $\alpha\beta = P$

we get, $x^2+6x+P=0$. — (2)

[We have relation $\alpha+\beta=-6$
$\alpha\beta = P$]

[then eq. becomes $x^2+6x+P=0$]

$\gamma i - \gamma i = 0$

$(\gamma i)(-\gamma i) = \gamma^2 = Q$

$x^2+Q=0$. — (3)

Multiply (2) & (3) & it become equal to given equation.

$(x^2+6x+P)(x^2+Q) = x^4+6x^3-16x^2+24x-80$

$x^4+6x^3+Px^2+Qx^2+6xQ+PQ = x^4+6x^3-16x^2+24x-80$

$P+Q=-16$

$6Q=24$

$\boxed{Q=4}$

$P+4=-16$

$\boxed{P=-20}$

$\gamma^2 = Q = 4$.

$\gamma = \pm 2i$

two roots are $+2i, -2i$.

$x^2+6x-20=0$.

$x = \dfrac{-6 \pm \sqrt{36-4(-20)}}{2}$

$= \dfrac{-6 \pm \sqrt{116}}{2}$

$= \dfrac{-6 \pm 2\sqrt{29}}{2}$

$\boxed{x = -3 \pm \sqrt{29}}$

Solve: $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$. Given that its roots in G.P.

let $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ are the roots of equation

$$f(x) = x^4 + 15x^3 + 70x^2 + 120x + 64 = 0.$$

Sum of roots $= -15$.

$$\frac{a}{r^3} + \frac{a}{r} + ar + ar^3 = -15$$

$$\frac{a + ar^2 + ar^4 + ar^6}{r^3} = -15$$

$$ar^6 + ar^4 + 15r^3 + ar^2 + a = 0.$$

$$r^2(ar^4 + ar^2 + 15r + a) + a = 0.$$

Product of roots $= 64$.

$$\frac{a}{r^3} \times \frac{a}{r} \times ar \times ar^3 = 64$$

$$a^4 = 64.$$

$$a^2 = 8$$

$$a = \pm\sqrt{8}.$$

Assume $\frac{a}{r^3} + ar^3 = P$ $\qquad$ $\frac{a}{r} + ar = Q.$

$\frac{a}{r^3} \cdot ar^3 = a^2$ $\qquad$ $\frac{a}{r} \cdot ar = a^2.$

$\boxed{a^2 = 8}$ $\qquad$ $\boxed{a^2 = 8}$

$x^2 - Px + 8 = 0$ $\qquad$ $x^2 - Qx + 8 = 0.$

$$(x^2 - Px + 8)(x^2 - Qx + 8) = x^4 + 15x^3 + 70x^2 + 120x + 64.$$

$$x^4 - Qx^3 + 8x^2 - Px^3 + PQx^2 - 8Px + 8x^2 - 8Qx + 64 = x^4 + 15x^3 \ldots$$

Compare Coefficient of x^3, x^2 & x.

$$-Q - P = 15$$

$$P + Q = -15 \quad -(1)$$

Coeff. of x^2.

$16 + PQ = 70$

$PQ = 70 - 16$

$PQ = 54$.

$\boxed{P = \frac{54}{Q}}$ — ②

Coeff. of x.

$-8(P+Q) = 120$

$P + Q = \frac{120}{-8}$

$P + Q = -15$

Put eq. ② in eq. ①

$\frac{54}{Q} + Q = -15$

$54 + Q^2 = -15Q$

$Q^2 + 15Q + 54 = 0$.

$Q^2 + 9Q + 6Q + 54 = 0$.

$Q(Q+9) + 6(Q+9) = 0$

$(Q+6)(Q+9) = 0$

$\boxed{Q = -6, -9}$ ✓

When $Q = -6$. ✓

then $P = -9$. ✓

$x^2 + 9x + 8 = 0$, $x^2 + 6x + 8 = 0$.

$x^2 + 8x + 1x + 8 = 0$, $x^2 + 4x + 2x + 8 = 0$

$x(x+8) + (x+8) = 0$, $x(x+4) + 2(x+4) = 0$.

$(x+1)(x+8) = 0$, $(x+4)(x+2) = 0$

$20x + 6$, $\boxed{x = -1, 8}$, $\boxed{x = -4, 2}$

When $Q = -9$

$P = -6$

$x^2 + 6x + 8 = 0$, $x^2 + 9x + 8 = 0$

$x = -4, -2$, $x = -1, -8$

So, The roots of equation are $-1, -2, -4, -8$.

$\frac{70}{16}$ 54

$8\overline{)120}$

2	54
3	27
3	9
3	3
	1

Q. Solve $x^4+2x^3-21x^2-22x+40=0$. Given that the diff of the roots is equal to the difference of other two roots.

∴ Let $\alpha, \beta, \gamma, \delta$ are root of given eq.

Acc. to Ques.

$\alpha-\beta = \gamma-\delta$

$\alpha+\delta = \beta+\gamma$ — (1).

$\alpha-\beta = \beta-\gamma$

Sum of roots = -2

$\alpha+\beta+\gamma+\delta = -2$ [using eq. (1)]

$2\beta+2\gamma = -2$

$\beta+\gamma = -1$

∴ $\alpha+\delta = -1$

Assume that

$\alpha\delta = P$ $\beta\gamma = Q$

$x^2+x+P=0$ — (*) $x^2+x+Q=0$ — (**)

Multiply (*) & (**) & compare with given equation

$(x^2+x+P)(x^2+x+Q) = x^4+2x^3-21x^2-22x+40.$

$x^4+x^3+Qx^2+x^3+x^2+Qx+Px^2+Px+PQ = x^4+2x^3-21x^2-22x+40$

$x^4+2x^3+(Q+P+1)x^2+(P+Q)x+PQ = x^4+2x^3-21x^2-22x+40$

Compare x^2, x & Constant.

$P+Q+1 = -21$

$P+Q = -22$ — (1)

$PQ = 40$

$P = \frac{40}{Q}$ — (2)

Put eq. (2) in eq (1)

DATE
PAGE

$$\frac{40}{Q} + Q = -22$$

$$40 + Q^2 + 22Q = 0.$$

$$Q^2 + 22Q + 40 = 0$$

$$Q^2 + 20Q + 2Q + 40 = 0$$

$$Q(Q+20) + 2(Q+20) = 0.$$

$$(Q+2)(Q+20) = 0.$$

$$\boxed{Q = -2, -20.}$$

When $Q = -2$
then $P = -20.$

$x^2 + x - 20 = 0,$

$$x = \frac{-1 \pm \sqrt{1+80}}{2}$$

$$= \frac{-1 \pm \sqrt{81}}{2}$$

$$= \frac{-1 \pm 9}{2}$$

$$= \frac{-1-9}{2}, \frac{-1+9}{2}$$

$$= -5, 4.$$

$x^2 + x - 2 = 0.$

$$x^2 + 2x - x - 2 = 0$$

$$x(x+2) - (x+2) = 0$$

$$(x-1)(x+2) = 0$$

$$x = 1, -2.$$

When $Q = -20$
then $P = -2$

$x^2 + x - 2 = 0,$ $\quad x^2 + x - 20 = 0.$

$x = 1, -2.$ $\quad x = -5, 4.$

$$\boxed{\begin{array}{l} 4 - 5 = -1 \\ 1 - 2 = -1 \end{array}}$$

The Roots of equation are $-5, -2, 1, 4.$

Q. Solve $x^4 - 2x^3 - 3x^2 - 4x - 1 = 0$ Given that the product of two roots is unity

$\therefore$ Let $\alpha, \beta, \gamma, \delta$ are the roots of equation

Given $\gamma\delta = 1$ — ①

Now Product of roots $= -1$

$\alpha\beta\gamma\delta = -1$ [using eq. ①].

$\alpha\beta = -1$

Assume $\alpha + \beta = P$, $\gamma + \delta = Q$

$x^2 - Px - 1 = 0$ — (*) $x^2 - Qx + 1 = 0$ — (**)

Multiply (*) & (**) & Compare with given equation

$(x^2 - Px - 1)(x^2 - Qx + 1) = x^4 - 2x^3 - 3x^2 - 4x - 1$

$x^4 - Qx^3 + x^2 - Px^3 + PQx^2 - Px - x^2 + Qx - 1 = x^4 - 2x^3 - 3x^2 - 4x - 1$

$x^4 + PQx^2 - (P+Q)x^3 + (Q-P)x - 1 = x^4 - 2x^3 - 3x^2 - 4x - 1$

Coeff. of x^3, x^2, x.

x^2 $PQ = -3$. — ①

x^3 $P + Q = 2$ — ②

x $P - Q = 4$.

Using ① & ② we get.

$x^2 - 2x - 3 = 0$.

$x^2 - 3x + x - 3 = 0$

$x(x-3) + (x-3) = 0$.

$(x+1)(x-3) = 0$

$\boxed{x = -1, 3}$

From eq ① $P = -3/Q$

Put P in eq. ②

$\frac{-3}{Q} + Q = 2$

$-3 + Q^2 = 2Q$

$Q^2 - 2Q - 3 = 0$

$\boxed{Q = -1, 3}$

Scanned with CamScanner

DATE
PAGE

Case 1:- When $Q = -1$
$P = 3$.

$x^2 - 3x - 1 = 0, \quad x^2 + x + 1 = 0.$

$x = \frac{+3 \pm \sqrt{9+4}}{2}, \quad x = \frac{-1 \pm \sqrt{1-4}}{2}$ $\quad -4(1)(-1)$

$= \frac{3 \pm \sqrt{13}}{2}, \quad = \frac{-1 \pm \sqrt{3}\,i}{2}$

Case 2:- When $Q = 3$
$P = -1$

$x^2 + x - 1 = 0, \quad x^2 - 3x + 1 = 0.$

$x = \frac{-1 \pm \sqrt{1+4}}{2}, \quad x = \frac{3 \pm \sqrt{9-4}}{2}$

$= \frac{-1 \pm \sqrt{5}}{2}, \quad = \frac{3 \pm \sqrt{5}}{2}$

So, the roots of equation are $\frac{3 \pm \sqrt{13}}{2}$, $\frac{-1 \pm \sqrt{3}\,i}{2}$ for $Q = -1, P = 3$.

the roots of equation are $\frac{-1 \pm \sqrt{5}}{2}$, $\frac{3 \pm \sqrt{5}}{2}$ for $Q = 3, P = -1$.

Q. Solve $x^4+15x^3+70x^2+120x+64=0$. Given that the roots are in G.P.

—— * —— * —— * —— * ——

Q. Solve $x^4+6x^3-16x^2+24x-80=0$. Given that sum of two of its roots is 0.

: Let $\alpha, \beta, \gamma, \delta$ are the roots of equation.

~~Given~~

$\alpha+\beta=0$

$(-1)^n \frac{a_n}{a_0}$

Sum of roots $=-6$

$x^2+P=0$

$\alpha+\beta+\gamma+\delta=-6$

$\gamma+\delta=-6$

Assume that $\alpha\beta=P$, $\gamma\delta=Q$

$x^2+P=0$ — (*) $x^2+6x+Q=0$ — (**)

Multiply (*) & (**) & Compare with given eq.

$(x^2+P)(x^2+6x+Q)=x^4+6x^3-16x^2+24x-80$

$x^4+6x^3+Qx^2+Px^2+6Px+PQ=x^4+6x^3-16x^2+24x-80$

Compare Coeff. of x^2 & x & constant.

$P+Q=-16$ — (1)

$6P=24$

$\boxed{P=4}$

Put P in eq. (1)

$4+Q=-16$

$\boxed{Q=-20}$

Put P & Q in eq. (*) & (**)

$x^2+4=0$

$x^2=-4$

$\boxed{x=\pm 2i}$

$x^2+6x-20=0$

$x=\frac{-6\pm\sqrt{36+80}}{2}$

$=\frac{-6\pm\sqrt{116}}{2}=\frac{-6\pm 2\sqrt{29}}{2}=-3\pm\sqrt{29}$

Q. Solve $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$, Given that two of its roots are equal in magnitude but opposite in sign.

:- let $\alpha, \beta, \gamma, \delta$ are the roots of given equation.

Given

$$\alpha = -\beta.$$

$$\alpha + \beta = 0$$

Sum of roots $= 8$

$$\alpha + \beta + \gamma + \delta = 8$$

$$\gamma + \delta = 8$$

Assume that $\alpha\beta = P$, $\gamma\delta = Q$

$$x^2 + P = 0 \quad -(*) \qquad\qquad x^2 - 8x + Q = 0 \quad -(**)$$

Multiply

$$(x^2 + P)(x^2 - 8x + Q) = x^4 - 8x^3 + 14x^2 + 8x - 15$$

$$x^4 - 8x^3 + Qx^2 + Px^2 - 8Px + PQ = x^4 - 8x^3 + 14x^2 + 8x - 15$$

Compare Coefficient of x^2, x, & Constant.

$$P + Q = 14.$$

$$-8P = 8 \Rightarrow \boxed{P = -1}$$

$$PQ = -15 \Rightarrow \boxed{Q = 15}$$

$x^2 - 14x - 15$

$x = \frac{14 \pm \sqrt{196 + 60}}{2}$

$\frac{14 \pm 16}{2}$

$\boxed{x = 15, -1}$

Put P & Q in eq. $(*)$ & $(**)$

$$x^2 - 1 = 0 \qquad\qquad x^2 - 8x + 15 = 0.$$

$$x^2 = 1 \qquad\qquad x = \frac{8 \pm \sqrt{64 - 60}}{2}$$

$$\boxed{x = \pm 1}$$

$$= \frac{8 \pm \sqrt{4}}{2}$$

$$= \frac{8 \pm 2}{2} = 4 \pm 1$$

$$\boxed{x = 5, 3}$$

The roots of equation are $-1, 1, 5, 3$.

* ——— * ——— *

UNIT : 2.

Transformation of Equations :-

i) Form an Equation Whose roots are negative of the roots of given equation.

: Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are roots of given eq.
We need to write an eq., Whose roots are $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$.

$\Rightarrow$ The relation between roots of new equation & old eq. is $\boxed{y = -x}$

If original eq. is $f(x) = 0$.
then transformed eq. is $f(-y) = 0$.

ii). Form an eq. Whose roots are m times those of the given eq.

: Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are roots of given eq.
We need to write an eq., Whose roots are $m\alpha_1, m\alpha_2, m\alpha_3, \ldots, m\alpha_n$.

$\Rightarrow$ The relation b/w roots of new eq. & old eq.

$y = mx. \Rightarrow \boxed{x = y/m}$

If original eq. is $f(x) = 0$.
then transformation eq is $f\left(\frac{y}{m}\right) = 0$.

(iii) Form an equation whose roots are reciprocal of those of the given eq.

Sol:- Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are roots of given eq.

we need to write an eq., whose roots are

$$\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \ldots, \frac{1}{\alpha_n}.$$

$\Rightarrow$ The Relation b/w roots of new equation & old eq. is

$$y = \frac{1}{x} \Rightarrow \boxed{x = \frac{1}{y}}$$

If original eq. is $f(x) = 0$.

then transformed eq. is $f\left(\frac{1}{y}\right) = 0$.

Example:- Form an equation, whose roots are roots of the equation $x^4 - 3x^2 + 7x - 1 = 0$ with their sign changed.

Sol: Put $x = -y$.

$(-y)^4 - 3(-y)^2 + 7(-y) - 1 = 0.$

$f(y) = y^4 - 3y^2 - 7y - 1 = 0.$

or

$f(x) = x^4 - 3x^2 - 7x - 1 = 0.$

(2) Transform the eq^n. $3x^4 - 4x^3 + 2x - 1 = 0$ into another with leading coefficient unity and remaining coeff. as integer.

Sol:- Given eq^n. is

$3x^4 - 4x^3 + 2x - 1 = 0,$

making the leading coeff. unity, we get.

$$x^4 - \frac{4}{3}x^3 + \frac{2}{3}x - \frac{1}{3} = 0.$$

$x^4 - \frac{4}{3}x^3 + 0x^2 + \frac{2}{3}x - \frac{1}{3} = 0.$

Multiply 3 with ~~…~~ powers from 0, 1, 2, 3, 4.

$3^0 x^4 - 3\,\frac{4}{3}x^3 + 3^2 \cdot 0x^2 + 3^3 \cdot \frac{2}{3}x - \frac{3^4}{3} = 0.$

$x^4 - 4x^3 + 18x - 27 = 0.$

Example: Remove the fractional coefficient, so that the leading Coeff. remains unity of the eq. $x^5 - \frac{1}{2}x^4 + \frac{5}{9}x^3 - \frac{3}{8}x + 6 = 0$

$\therefore\ x^5 - \frac{1}{2^1 3^0}x^4 + \frac{5}{2^0 3^2}x^3 + \frac{0}{2^0 3^0}x^2 - \frac{3}{2^3 3^0}x + \frac{6}{2^0 3^0} = 0$ $\qquad 2^1 3^0$

$m = 2^r 3^s$

Powers of 2 are in order 1, 0, 0, 3, 0.

Divide them by 1, 2, 3, 4, 5

we get, $\quad 1, 0, 0, \frac{3}{4}, 0.$

$r = 1.$

Powers of 3 are in order 0, 2, 0, 0, 0.

Divide them by $0, \frac{2}{2}, 0, 0, 0.$

We get, $\quad 0, 1, 0, 0, 0$ $\qquad 1.5,\ 2,\ 3,\ 8.5$

$s = 1$

& $m = 2^1 \times 3^1$

$= 6.$

$$6^4x^5 - \frac{6}{2}x^4 + \frac{36\times5x^3}{9} - \frac{(6^4)3x}{8} + (6^4)^6 = 0.$$

$$x^5 - 3x^4 + 20x^3 - 486x + 46656 = 0.$$

18 × 6
36
84
20 × 6
27 × 9
162
3
486
6 × 20 × 6

Example:
Solve $6x^3 - 11x^2 + 6x - 1 = 0$, given that the roots are in H.P.

Let α, β, γ are the roots of given eq.

$\boxed{a, b, c \text{ in H.P. then } \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ in A.P.}}$

$\Rightarrow \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in A.P.

Construct an eq. whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.
Put $x = \frac{1}{y}$.

$$6\left(\frac{1}{y}\right)^3 - 11\left(\frac{1}{y}\right)^2 + 6\left(\frac{1}{y}\right) - 1 = 0.$$

$$\frac{6}{y^3} - \frac{11}{y^2} + \frac{6}{y} - 1 = 0.$$

$$6 - 11y + 6y^2 - y^3 = 0.$$

$\Rightarrow y^3 - 6y^2 + 11y - 6 = 0.$ $(-1)^n \frac{a_n}{a_0}$

Let $a-d, a, a+d$ are roots of new eq.
where $a-d = \frac{1}{\alpha}$, $a = \frac{1}{\beta}$, $a+d = \frac{1}{\gamma}$.

Sum of roots $= +6$.

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 6.$$

$$\frac{\gamma\beta + \gamma\alpha + \beta\alpha}{\alpha\beta\gamma} = 6$$

Product of roots $=$

$$\frac{1}{\alpha\beta\gamma} = 6$$

$$6\alpha\beta\gamma = 1.$$

$\alpha\beta+\beta\gamma+\gamma\alpha = 6\alpha\beta\gamma.$

$\alpha\beta+\beta\gamma+\gamma\alpha = 1$

$\Rightarrow$ $y^3-6y^2+11y-6=0.$

Let $a-d, a, a+d$ are the roots of above eq.

or $a-d = \frac{1}{\alpha}$, $a = \frac{1}{\beta}$, $a+d = \frac{1}{\gamma}.$

Sum of roots $=6.$

$a-d+a+a+d=6$

$3a=6$

$\boxed{a=2}$

Product of roots $=6$

$(a-d)(a+d)a=6$

$(a^2-d^2)a=6$

$(4-d^2)2=6$

$4-d^2=3.$

$-d^2=3-4$

$d^2=1$

$\boxed{d=\pm 1}$

When $d=1$

$\frac{1}{\alpha}=a-d=1 \Rightarrow \alpha=1$

$\frac{1}{\beta}=a=2 \Rightarrow \beta=\frac{1}{2}.$

$\frac{1}{\gamma}=a+d=3 \Rightarrow \gamma=\frac{1}{3}.$

When $d=-1$

$\frac{1}{\alpha}=a-d=3 \Rightarrow \alpha=3.$

$\frac{1}{\beta}=a=2 \Rightarrow \beta=\frac{1}{2}$

$\frac{1}{\gamma}=a+d=1 \Rightarrow \gamma=1.$

Roots of given equation is $1, \frac{1}{2}, \frac{1}{3}.$

DATE
PAGE

Exercise 8: Find & solve $40x^4 + \lambda x^3 - 2x^2 - 2x + 1 = 0$ given that the roots are in H.P

Sol:

Ex: Diminish the roots of equation $2x^5 - x^3 + 10x - 8 = 0$ by 5.

	x^5	x^4	x^3	x^2	x	c
5	2	0	-1	0	10	-8.
		10	50	245	1225	6125
5	2	10	49	245	1235	$\boxed{6167.}$
		10	100	745	4950	
5	2	20	149	990	$\boxed{6185}$	
		10	150	1495		
5	2	30	299	$\boxed{2485}$		
		10	200			
5	2	40	$\boxed{499}$			
		10				
	2	$\boxed{50}$				

$$2y^5 + 50y^4 + 499y^3 + 2485y^2 + 6185y + 6167 = 0.$$

Ex: Increase the roots of equation $3x^5 - 5x^3 + 7 = 0$ by 4.
*

	x^5	x^4	x^3	x^2	x	c
-4	3	0	-5	0	0	7
		-12	48.	-172	688	-2752
-4	3	-12	43.	-172	688	$\boxed{-2745}$
		-12	96	-556	2912.	
-4	3	-24	139	-728	$\boxed{3600}$	
		-12	-144	20.		
-4	3	-36	-5	$\boxed{-708}$		
		-12	192			
-4	3	-48	$\boxed{187}$			
		-12				
	3	-60				

$\alpha+\beta+\gamma+\delta+\eta=0$

$\alpha-5+\beta-5+\gamma-5+\delta-5+\eta-5$

$x=-4$

Example:- Remove the second term from eq^n $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$ & hence sol^n the equation.

Sol:-

	x^4	x^3	x^2	x	c
+4	1	-16	86	-176	105
		+4	-48	152	-96
4	1	-12	38	-24	$\boxed{9}$ c.
		4	-32	24	
4	1	-8	6	$\boxed{0}$ y	
		4	-16		
4	1	-4	$\boxed{-10}$ y^2		
		4			
	1 y^4	0 y^3			

$y^4 - 10y^2 + 9 = 0.$ ✓ $\pm 1, \pm 3$

$y^2 = z$

$z^2 - 10z + 9 = 0.$

$z^2 - 9z - z + 9 = 0.$

$z = 9, 1.$

$y^2 = 1, 9.$

$\boxed{y = \pm 1, \pm 3.}$

$x = 4 \pm 1, 4 \pm 3$
$= 5, 3, 7, 1.$

Example:- Remove the third term from eq^n $x^4 - 4x^3 - 18x^2 - 3x + 2 = 0.$

Suppose decrease each root by h, so that new eq^n does not have 3rd term.

	x^4	x^3	x^2	x	C
h	1	-4	-18	-3	2
		h	h^2-4h	h^3-4h^2-18h	$h^4-4h^3-18h^2-3h$
h	1	$h-4$	$h^2-4h-18$	$h^3-4h^2-18h-3$	$\boxed{h^4-4h^3-18h^2-3h+2}$
		h	$2h^2-4h$	$3h^3-8h^2-18h$	
h	1	$2h-4$	$3h^2-8h-18$	$\boxed{4h^3-12h^2-36h-3}$	
		h	$3h^2-4h$		
h	1	$3h-4$	$\boxed{6h^2-12h-18}$		
		h			
	1	$\boxed{4h-4}$			

$1+4-18+3+2$
$=-8$
$-4-12+36-3$
$=17$
$6+12-18$
$=0$
-8

$6h^2-12h-18=0.$

$6(h^2-2h-3)=0.$

$h^2-2h-3=0.$ When $h=-1$

$h^2-3h+h-3=0.$ $y^4-8y^3+17y-8=0.$

$(h-3)(h+1)=0$

$h=-1,3.$

Exercise :- Show that the same transformation removes both 2nd & 4th term of $x^4+16x^3+83x^2+152x+84=0$. Hence solve.

(2) Reduce the equation $2x^3-3x^2+10x-4=0$ to the form $z^3+3Hz+G=0$, where H & G are constants. Find H & G.

(3) Transform the eq. $x^3-6x^2+11x-6=0$ by the transformation $y=x-2$. Hence, find all roots of given eq^n.

Method to form an equation, whose roots are squares of the roots of given eq.

Put $y = x^2$.

[in Eq. where x^2 is present we have to put y in place of x^2.]

Example:-

$x^3 - x^2 - 2x + 2 = 0$.

$yx - y - 2x + 2 = 0$ [Put $y = x^2$]

$1 - 1 - 2 + 2 = 0$

~~$y(x-1) = 2(x-1)$~~

$xy - 2x = y - 2$.

$(-\sqrt{2})^3$

$-2\sqrt{2} - 2 + 2\sqrt{2} + 2$

$x(y-2) = y - 2$

Squaring

$x^2(y-2)^2 = (y-2)^2$.

$y(y-2)^2 = (y-2)^2$.

$\Rightarrow y(y-2)^2 - (y-2)^2 = 0$.

$(y-2)^2(y-1) = 0$

$y = 1, 2, 2$.

$y[y^2 - 4y + 4] = y^2 - 4y + 4$

So given equation's roots are $1, \sqrt{2}, -\sqrt{2}$.

$y^3 - 4y^2 + 4y - y^2 + 4y - 4 = 0$.

to check the sign of roots we have to apply Descartes rule & then repeated root theorem

$y^3 - 5y^2 + 8y - 4 = 0$.

Find an Equation whose roots are the cubes of roots of the equation.

$x^4 - x^3 + 2x^2 + 3x + 1 = 0$.

Put $y = x^3$.

$yx - y + 2x^2 + 3x + 1 = 0$.

$\cancel{x(y+3) = y - 2x^2 - 1}$

$= a^3 + b^3 + c^3 + 2ab + 2bc + 2ca$

$xy - y = -3x - 2x^2 - 1$

Cubing both side

$(xy - y)^3 = -(3x + 2x^2 + 1)^3$

$x^3y^3 - y^3 - 3x^2y^3 + 3xy^3 = -[27x^3 + 8x^6 + 1 + 12x^3 + 4x^2 + 6x]$

$x^3y^3 - y^3 - \underline{3x^2}y^3 + \underline{3xy^3} = -27x^3 - 8(x^3)^2 - 1 - 12x^3 - 4x^2 - 6x$

$y^4 - y^3 - 3x^2y^3 + 3xy^3 = -27y - 8y^2 - 1 - 12y - \underline{4x^2 - 6x}$

$y^4 - y^3 + 8y^2 + 39y + 1 = -4x^2 + 3x^2y^3 - 6x - 3xy^3$

-27
-12
$\overline{39}$

$xy + 3x + 2x^2 = y - 1$

$x(y + 3 + 2x) = y - 1$

Cubing both side

$x^3[(y+3) + 2x]^3 = (y-1)^3$

$x^3[(y+3)^3 + 8x^3 + 3(y+3)^2(2x) + 3(y+3)(2x)^2] = (y-1)^3$

$x^3[(y+3)^3 + 8x^3 + 6x(y+3)^2 + 12x^2(y+3)] = (y-1)^3$

$x^3[(y+3)^3 + 8x^3 + x[y+3][6(y+3) + 12x(y+3)]] = (y-1)^3$

$y[(y+3)^3 + 8x^3 + 6x(y+3)[y+3+2x]] = (y-1)^3$

$y\left[(y+3)^3 + 8x^3 + 6\cancel{x}(y+3)\frac{(y-1)}{x}\right] = (y-1)^3$

$y[(y+3)^3 + 8x^3 + 6(y+3)(y-1)] = (y-1)^3$

Scanned with CamScanner

$a_0 - a_1 + a_2 -$

$\frac{a_2}{a_0}$

DATE
PAGE

Example:-

Find an equation whose roots are square of the roots of the eqⁿ $x^4 + bx^2 + cx + d = 0$ - ①

Put $x^2 = y$.

$$y^2 + by + cx + d = 0.$$

$$y^2 + by + d = -cx$$

squaring

$$(y^2 + by + d)^2 = c^2 x^2.$$

$$y^4 + (by + d)^2 + 2y^2(by + d) = c^2 y$$

$$y^4 + 2by^3 + (b^2 + 2d)y^2 + (2bd - c^2)y + d^2 = 0. \quad -②$$

Hence show that if α, β, γ & δ are four no. s.t. $\sqrt{\gamma} + \sqrt{\beta} + \sqrt{\delta} + \sqrt{\gamma} = 0$, then $64\alpha\beta\gamma\delta - [4\Sigma\alpha\beta - (\Sigma\alpha)^2]^2 = 0$

let $\alpha, \beta, \gamma, \delta$ be roots of eqⁿ ②. Then $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}, \sqrt{\delta}$ are roots of eq. ① s.t. $\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} + \sqrt{\delta} = 0$.

$$\alpha\beta\gamma\delta = \text{Product of roots of ②.}$$
$$= d^2.$$

$\Sigma\alpha\beta = (b^2 + 2d)$. [

$\Sigma\alpha = -2b$ [Sum of roots of eq. ②]

Now, $64\alpha\beta\gamma\delta - [4\Sigma\alpha\beta - (\Sigma\alpha)^2]^2 = 0$.

$$64d^2 - [4(b^2 + 2d) - 4b^2]^2 = 0$$

$$64d^2 - [4b^2 + 8d - 4b^2]^2 = 0$$

$$64d^2 - 64d^2 = 0$$

$$0 = 0.$$

(1) If α, β, γ are roots of $x^3 - 12x^2 + 39x + 8 = 0$, find an equation whose roots are $\frac{1}{\alpha-2}, \frac{1}{\beta-2}, \frac{1}{\gamma-2}$.

Put $y = \frac{1}{x-2}$

$\frac{1}{y} = x - 2$

$\boxed{\frac{1}{y} + 2 = x}$

$\frac{1+2y}{y} = x.$

$\left(\frac{1+2y}{y}\right)^3 - 12\left(\frac{1+2y}{y}\right)^2 + \frac{39}{y}(1+2y) - 8 = 0$

$\frac{1}{y^3}[1 + 8y^3 + 6y + 12y^2] - \frac{12}{y^2}(1 + 4y^2 + 4y] + \frac{39}{y}(1+2y) - 8 = 0$

$\frac{1}{y^3} + 8 + \frac{6}{y^2} + \frac{12}{y} - \frac{12}{y^2} - 48 - \frac{48}{y} + \frac{39}{y} + 78 - 8 = 0.$

$\frac{1}{y^3} - \frac{6}{y^2} + \frac{3}{y} + 46 = 0.$

~~$\frac{46y^3 + 47y^2 - 6y + 1}{y^3} = 0$~~. $\quad 30y^3 + 3y^2 - 6y + 1 = 0.$

~~$8y^3 + 42y^2 - 6y + 1 = 0$~~

12
39
51

(3) If α, β, γ are roots of $x^3 + ax^2 + bx + c = 0$, form an eq. whose roots are $\frac{\alpha}{\beta+\gamma}, \frac{\beta}{\gamma+\alpha}, \frac{\gamma}{\gamma+\beta}$

$$\frac{\alpha}{\beta+\gamma} = \frac{\alpha}{\underbrace{\beta+\gamma+\alpha}-\alpha} = \frac{\alpha}{-a-\alpha} = \frac{-\alpha}{a+\alpha}$$

$$\frac{\beta}{\gamma+\alpha} = \frac{-\beta}{a+\beta}$$

$$\frac{\gamma}{\alpha+\beta} = \frac{-\gamma}{a+\gamma}.$$

Put $y = \frac{-x}{a+x}.$

$(a+x)y = -x.$ $\qquad ay + xy + x = 0$

$\qquad x(y+1) = -ay$

$x(y+1) = -ay$

$x = -\frac{ay}{1+y}.$

$$\left(-\frac{ay}{1+y}\right)^3 + a\left(-\frac{ay}{1+y}\right)^2 + b\left(\frac{-ay}{1+y}\right) + c = 0.$$

$$\frac{-a^3y^3}{1+y^3+3y+3y^2} + \frac{a^3y^2}{(1+y^2+2y)} - \frac{aby}{(1+y)} + c = 0.$$

$$\frac{-a^3y^3 + a^3y^2(1+y) - aby(1+y)^2 + c(1+y)^3}{(1+y)^3} = 0$$

$$-\cancel{a^3y^3} + a^3y^2 + \cancel{a^3y^3} - aby[1+y^2+2y] + c[1+y^3+3y+3y^2] = 0$$

$$a^3y^2 - aby - aby^3 - 2aby^2 + c + cy^3 + c3y + c3y^2 = 0.$$

DATE
PAGE

$y^3(c-ab)+y^2(a^2+3c-2ab)+y(3c-ab)+c=0.$

P/w. If, α, β, γ are roots of the eq^n. $x^3-ax^2+bx-c=0$
Find an equation whose roots are $\beta\gamma - \frac{1}{\alpha}$

S.

$$\frac{\alpha\beta\gamma}{\alpha}-\frac{1}{\alpha}=\frac{c}{\alpha}-\frac{1}{\alpha}=\frac{c-1}{\alpha}$$

$$=\frac{c-1}{\beta}$$

$$=\frac{c-1}{\gamma}.$$

$$y=\frac{c-1}{x}.$$

Put $\boxed{x=\frac{c-1}{y}}$

$$\frac{(c-1)^3}{y^3}-\frac{a(c-1)^2}{y^2}+\frac{b(c-1)}{y}-c=0.$$

$$(c-1)^3-a(c-1)^2y+b(c-1)y^2-cy^3=0.$$

$$cy^3-b(c-1)y^2+a(c-1)^2y-(c-1)^3=0.$$

DATE
PAGE

Exp. If α, β, γ are roots of $x^3 + ax + b = 0$ (1) $(b \neq 0)$, then form an equation whose roots are $\beta^2 + \beta\gamma + \gamma^2$, $\gamma^2 + \gamma\alpha + \alpha^2$, $\alpha^2 + \alpha\beta + \beta^2$.

: Let α, β, γ are the roots of given eqn.

$$\beta^2 + \beta\gamma + \gamma^2 = (\beta+\gamma)^2 - \beta\gamma$$

$$= (\alpha+\beta+\gamma-\alpha)^2 - \frac{\alpha\beta\gamma}{\alpha}$$

$$= (0-\alpha)^2 + \frac{b}{\alpha}$$

$$= \alpha^2 + \frac{b}{\alpha}, \beta^2 + \frac{b}{\beta}, \gamma^2 + \frac{b}{\gamma}$$

$$y = x^2 + \frac{b}{x}$$

Put. $y = \frac{x^3 + b}{x}$

$x = \frac{x^3 + b}{y}$

$xy - x^3 - b = 0$ — (*)

$x(y - x^2) = b$

$x = b$ or $y - x^2 = b$

Add eq. (1) & (*)

$$\cancel{x^3} + ax + \cancel{b} = 0$$
$$-\cancel{x^3} + xy - \cancel{b} = 0$$

$(a+y)x = 0$

$x = 0$, $a + y = 0$

not possible

$y + a = 0 \Rightarrow y^3 + a^3 = 0$

Cubing b.s.

$(y+a)^3 = 0$

$y^3 + a^3 + 3a^2y + 3ay^2 = 0$

$$\boxed{y^3 + 3ay^2 + 3a^2y + a^3 = 0}$$

$y^3 = -a^3$.
$y^3 + a^3 = 0$.

PAGE

(1)

(Q) If α, β, γ are roots of $x^3+1=0$. Form an eq^n. whose roots are $\frac{\alpha}{(\beta-\gamma)^2}, \frac{\beta}{(\gamma-\alpha)^2}, \frac{\gamma}{(\alpha-\beta)^2}$. Hence Evaluate $\sum \frac{\alpha}{(\beta-\gamma)^2}$.

S: $x^3+1=0$

$x^3=-1$

$x=-1$

$x^2=1$

Relation of roots.

$$\frac{\alpha}{(\beta-\gamma)^2} = \frac{\alpha}{\beta^2+\gamma^2-2\beta\gamma+2\beta\gamma-2\beta\gamma}$$

$\alpha^2+\beta^2$

$$= \frac{\alpha}{(\beta+\gamma)^2-4\beta\gamma}$$

$$= \frac{\alpha}{(-\alpha)^2-\frac{4\beta\gamma\alpha}{\alpha}} = \frac{\alpha}{\alpha^2+\frac{4}{\alpha}}$$

$$= \frac{\alpha^2}{\alpha^3+4}$$

$$y = \frac{x^2}{x^3+4}$$

$x^3y+4y=x^2$.

$x^3y+4y-x^2=0$. — (2)

Eq. (2).

$x^3y-x^2+4y=0$.

$-y-1+4y=0$.

$(3y-1)^2=0$

$-y+4y=$

$3y=1$

$27y^3=1$

$27y^3-1=0$

$(3y-1)^3=0$.

$27y^3-1-27y^2+9y=0$.

~~$3y-1=0$~~

~~$(3y-1)^3=0$~~

~~$27y^3-1-27y^2+9y=0$~~

3abc = 0.

DATE
PAGE

Ex. If α, β, γ are roots of $x^3 - x - 1 = 0$. prove that

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = -7.$$

$$y = \frac{1+x}{1-x}.$$

$y - xy = 1 + x.$

$y - 1 = x + xy.$

$y - 1 = x(y+1)$

$$\frac{\cancel{1}}{x} = \frac{y+1}{y-1}$$

$$x = \frac{y-1}{y+1}.$$

$$\frac{(y-1)^3}{(y+1)^3} - \frac{(y-1)}{(y+1)} - 1 = 0.$$

$$\frac{(y-1)^3 - (y-1)(y+1)^2 - (y+1)^3}{(y+1)^3} = 0.$$

$y^3 - 1 - 3y^2 + 3y - (y-1)[y^2 + 1 + 2y] - [y^3 + 1 + 3y^2 + 3y] = 0.$

$\cancel{y^3} - 1 - 3y^2 + \cancel{3y} - \cancel{y^3} - 1 - 3y^2 - \cancel{3y} - [y^3 + y + 2y^2 - y^2 - 1 - 2y].$

$= -2 - 6y^2 - y^3 - y - 2y^2 + y^2 + 1 + 2y.$

$= -y^3 - 7y^2 - y - 1 = 0$

$y^3 + 7y^2 - y + 1 = 0.$

DATE
PAGE

(3)

Ex:- If α, β, γ are roots of the eqⁿ: $x^3+3x+2=0$ find an equation whose roots are $(\alpha-\beta)(\alpha-\gamma)$, $(\beta-\gamma)(\beta-\alpha)$, $(\gamma-\alpha)(\gamma-\beta)$.

Sol:- $(\alpha-\beta)(\alpha-\gamma) = \alpha^2-\alpha\gamma-\alpha\beta+\beta\gamma$
$= \alpha^2-(\alpha\beta+\alpha\gamma)+\beta\gamma$
$= \alpha^2-(\alpha\beta+\alpha\gamma+\beta\gamma)+2\beta\gamma$
$= \alpha^2-3+2\beta\gamma$
$= \alpha^2-3+\frac{2\beta\gamma\alpha}{\alpha}$

$= \alpha^2-3-\frac{4}{\alpha}$

$= \frac{\alpha^3-3\alpha-4}{\alpha}$

$x(3+3^{..})$

Put $y = \frac{x^3-3x-4}{x}$

$xy = x^3-3x-4$

$0 = x^3-(3+y)x-4$ — (2)

$\frac{3}{3x}$

$x^3+3x+2=0$
$x^3-(3+y)x-4=0$
$3+(3+y)x+6=0$

$(6+y)x=-6$
$x=\frac{-6}{6+y}$

$\frac{-216}{(6+y)^3}-\frac{18}{(6+y)}+2=0$

$\frac{-216-18(6+y)^2+2(6+y)^3}{(6+y)^3}=0$

$2(6+y)^3-18(6+y)^2-216=0$

$2[216+y^3+18y^2+108y]-18[36+y^2+12y]-216=0$

DATE
PAGE

$432 + 2y^3 + 36y^2 + 216y - 648 - 18y^2 -$

216 2 432 1 648 108 216 36 18 27 36 648 18 12

Let α, β, γ be roots of the eqⁿ $x^3 + 2x^2 + 3x + 1 = 0$. Form an eq. whose roots are $\frac{1}{\beta^3} + \frac{1}{\gamma^3} - \frac{1}{\alpha^3}$,

$\frac{1}{\alpha^3} + \frac{1}{\beta^3} - \frac{1}{\gamma^3}, \frac{1}{\beta^3} + \frac{1}{\gamma^3} - \frac{1}{\alpha^3}$.

Form an eq. whose roots are $\alpha^3, \beta^3, \alpha^3$.

Put $y = x^3$.

$y + 2x^2 + 3x + 1 = 0$.

$y + 1 = -2x^2 - 3x$

$(y+1) = -x(2x+3)$

Cube

$(y+1)^3 = -y(2x+3)^3$.

$y^3 + 1 + 3y^2 + 3y = -y[8y + 27 + 54x + 36x^2]$.

$= -y[8y + 27 + 18x(3 + 2x)]$.

$9x^2$

$y^3 + 1 + 3y^2 + 3y = -y\left[8y + 27 - 18(y+1)\right]$ $-y(-10y + 18)$

$-18y(y-1)$

$y^3 + 1 + 3y^2 + 3y = -8y^2 - 27y + 18y^2 + 18y$

27
18
45

$y^3 + 3y^2 - 10y^2 + 3y + 9y + 1 = 0$

$y^3 - 7y^2 + 12y + 1 = 0$ — (1)

$y^3 - 7y^2 - 1 = 10y^2$

27
-18
9

Now we Construct an eq. whose roots are $\frac{1}{B} + \frac{1}{C} - \frac{1}{A}$

$\frac{1}{C} + \frac{1}{A} - \frac{1}{B}$

$\frac{1}{A} + \frac{1}{B} - \frac{1}{C}$

$\frac{A - A + B + C}{BC} - \frac{1}{A}$

$\frac{7 - A}{\frac{-1}{A}} - \frac{1}{A}$

$\frac{7A - A^2}{-1} - \frac{1}{A}$

$A^2 - 7A - \frac{1}{A}$

$A(A-7) - \frac{1}{A}$

Put $z = y(y-7) - \frac{1}{y}$

$y = \frac{x^3 - 7x^2 - 1}{x}$

$yx = x^3 - 7x^2 - 1$

$= x^3 - 7x^2 - y$

DATE
PAGE

$$yz = (y-7)y^2 - 1.$$

$\Rightarrow y^3 - 7y^2 - yz - 1 = 0. \quad -(2).$

① - ②

$12y + 1 + yz + 1 = 0.$

$12y + yz + 2 = 0$

$y(12+z) = -2$

$y^3(12+z)^3 = -8.$

$$= \frac{A^3 - 7A^2 - 1}{A}$$

$$z = \frac{y^3 - 7y^2 - 1}{y}$$

$$z = \frac{-12y - 1 - 1}{y}$$

$yz = -12y - 2.$

$$z = \frac{-12y - 2}{y}.$$

$$y^3 - 7y^2 - y\left(\frac{-(12y+2)}{y}\right) - 1 = 0$$

$y^3 - 7y^2 + 12y + 2 - 1 = 0$

$y^3 - 7y^2 + 12y + 1 = 0.$

$zy = -12y - 2$

$zy + 12y = -2$

$y(z+12) = -2$

$\boxed{y = \frac{-2}{z+12}}$ Put in eq. ③.

$$\frac{(-2)^3}{(z+12)^3} - \frac{7(-2)^2}{(z+12)^2} + \frac{12(-2)}{(z+12)} + 1 = 0.$$

$$\frac{-8 - 28(z+12) - 24(z+12)^2 + (z+12)^3}{(z+12)^3} = 0$$

$$(z)^3 + (12)^3 + 36z^2 + 3z(144) - 24(z^2 + 144 + 24z) - 28z - 336 - 8$$

$$z^3 + (36-28)z^2 + (432 - 28 - 576)z - 24(144) + (12)^3 - 344$$

$$\cancel{z^3 + 12z^2 + 158z - 3624 + 1728 = 0.}$$

$$z^3 + 8z^2 - 172z - 3456 + 1728 - 344 = 0$$

$$z^3 + 8z^2 - 172z - 2072 = 0$$

Symmetric Function of roots.

Def. A function of roots of an equation is called symmetric function, if its value remains unchanged, when any roots of the equation are interchanged.

Example: let α, β, γ are the roots of eq^n $x^3+bx^2+cx+d=0$ then $f(\alpha,\beta,\gamma) = \alpha+\beta+\gamma$.

Example (i) Consider the equation $x^3+2x^2+3x+1=0$. Evaluate $\Sigma\alpha^2\beta$

: Let α,β,γ are the roots of given eq^n.

$$\Sigma\alpha^2\beta = \alpha^2\beta+\beta^2\gamma+\gamma^2\alpha+\beta^2\alpha+\gamma^2\beta+\alpha^2\gamma$$

$\alpha+\beta+\gamma=-2$

$\alpha+\beta=-2-\gamma$

$\alpha\beta\gamma=-1$

$$= \alpha\beta(\alpha+\beta)+\gamma\beta(\beta+\gamma)+\alpha\gamma(\gamma+\alpha)$$
$$= \alpha\beta(-2-\gamma)+\gamma\beta(-2-\alpha)+\alpha\gamma(-2-\beta)$$
$$= -2\alpha\beta-\alpha\beta\gamma-2\gamma\beta-\alpha\gamma\beta-2\alpha\gamma-\alpha\gamma\beta$$
$$= -2\alpha\beta+1-2\gamma\beta+1-2\alpha\gamma+1$$
$$= -2(\alpha\beta+\gamma\beta+\alpha\gamma)+3$$
$$= -6+3$$
$$= -3.$$

(ii) Find $\Sigma\alpha^2$.

$$\Sigma\alpha^2 = \alpha^2+\beta^2+\gamma^2$$
$$= \alpha.\alpha+\beta.\beta+\gamma.\gamma$$
$$= \frac{\alpha\beta\gamma}{\beta\gamma}+\frac{\alpha\beta\gamma}{\alpha\gamma}+\frac{\alpha\gamma\beta}{\beta\alpha}$$
$$= -2.$$

$\alpha^2+\beta^2+\gamma^2+2\alpha\gamma+2\alpha\beta+2\gamma\beta$

$(\alpha+\beta+\gamma)^2-2(3)$

$=4-6$

$=-2$

$$\Sigma\alpha^2 = \alpha^2+\beta^2+\gamma^2.$$
$$= \alpha^2+\beta^2+\gamma^2+2\alpha\beta-2\alpha\beta+2\beta\gamma-2\beta\gamma+2\gamma\alpha-2\gamma\alpha$$
$$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha).$$
$$= (-2)^2 - 2(3)$$
$$= 4-6$$
$$= -2.$$

Example iii) $\Sigma\alpha^3\beta = \alpha^3\beta+\beta^3\gamma+\gamma^3\alpha+\beta^3\alpha+\alpha^3\gamma+\gamma^3\beta.$

$\cancel{\alpha^3+\beta^3+\gamma^3+3\alpha\beta\gamma-3\alpha\beta\gamma}$.

$$= \alpha^3(\beta+\gamma)+\beta^3(\gamma+\alpha)+\gamma^3(\alpha+\beta)$$
$$= \alpha^3(-2-\alpha)+\beta^3(-2-\beta)+\gamma^3(-2-\gamma).$$

$$= -2\alpha^3-\alpha^4-2$$

$$= \alpha\beta(\alpha^2+\beta^2)+\beta\gamma(\beta^2+\gamma^2)+\gamma\alpha(\gamma^2+\alpha^2)$$
$$= \alpha\beta(-2-\gamma^2)+\beta\gamma(-2-\alpha^2)+\gamma\alpha(-2-\beta^2).$$
$$= -2\alpha\beta-\alpha\beta\gamma^2-2\beta\gamma-\alpha^2\beta\gamma-2\alpha\gamma-\beta^2\gamma\alpha$$
$$= -2(3)+\gamma+\beta+\alpha$$
$$= -6-2$$
$$= -8.$$

(4) $\Sigma\alpha^2\beta\gamma. = \alpha^2\beta\gamma+\beta^2\alpha\gamma+\gamma^2\beta\alpha.$
$$= \alpha\beta\gamma(\alpha+\beta+\gamma)$$
$$= -1(-2)$$
$$= 2.$$

H.W. v) $\Sigma \alpha^2\beta^2$ 5

vi) $\Sigma \frac{\beta\gamma}{\alpha}$ -7

vii) $\Sigma \frac{\beta^2+\gamma^2}{\beta\gamma}$ 3.

viii) $\Sigma \alpha^2\beta^2$

$\frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\alpha^2+\beta^2}{\beta\alpha} + \frac{\gamma^2+\alpha^2}{\alpha\gamma}$

$\cancel{\Sigma\alpha^2\beta^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}$

$\cancel{= \alpha^2(\beta^2+\gamma^2)+\beta^2\gamma^2}$

$\cancel{= \alpha^2(-2-\alpha^2)+\beta^2\gamma^2}$

$\cancel{= -2\alpha^2-\alpha^3+\frac{\beta^2\gamma^2\alpha^2}{\alpha^2}}$

$\frac{(-1)^2}{\alpha^2}$

v) $\Sigma\alpha^2\beta^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$

Multiply & Divide with $(\alpha\beta\gamma)^2$.

$\cancel{= \frac{(\alpha\beta\gamma)^2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)}{(\alpha\beta\gamma)^2}}$

$\cancel{= \frac{\alpha^4\beta^4\gamma^2+\beta^4\gamma^4\alpha^2+\gamma^4\alpha^4\beta^2}{(\alpha^2\beta^2\gamma^2)}}$

$\cancel{= \frac{\alpha^2\beta^2\gamma^2}{\gamma^2}+\frac{\beta^2\gamma^2\alpha^2}{\alpha^2}+\frac{\gamma^2\alpha^2\beta^2}{\beta^2}}$

$\cancel{= \frac{(-1)^2}{\gamma^2}+\frac{(-1)^2}{\alpha^2}+\frac{(-1)^2}{\beta^2}}$

$\cancel{= \frac{1}{\gamma^2}+\frac{1}{\alpha^2}+\frac{1}{\beta^2}}$

$\frac{1}{\gamma^2}+\frac{1}{\beta^2}+$

$\frac{-2}{1} \cdot -2.$

$\frac{\beta\gamma}{\alpha}+\frac{\gamma\alpha}{\beta}+\frac{\alpha\beta}{\gamma}$

$\frac{\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2}{\alpha\beta\gamma}$

$\frac{\beta^2\gamma^2+\alpha^2\gamma^2+\beta^2\gamma^2}{\alpha\beta\gamma}$

v) $\Sigma\alpha^2\beta^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$.

~~From part ii) $\Sigma\alpha^2\beta = -3$~~

~~$= \alpha^2\beta\cdot\beta+\beta^2\gamma\cdot\gamma+\gamma^2\alpha\cdot\alpha$~~

We have eq $x^3+2x^2+3x+1=0$. Whose roots are α, β, γ.
Now, we have to form an eq. Whose roots are $\alpha^2, \beta^2, \gamma^2$.

Put $x^2 = y$.

$yx+2y+3x+1=0$.

$x(y+3) = -(1+2y)$.

Squaring

$x^2(y+3)^2 = (-1)^2(1+2y)^2$

$y[y^2+9+6y] = (1+4y^2+4y)$.

$y^3+9y+6y^2 = 1+4y^2+4y$.

$y^3+2y^2+5y-1=0$.

$\therefore \Sigma\alpha^2\beta^2 = \frac{a_2}{a_0} = 5$.

vi) $\Sigma\frac{\beta\gamma}{\alpha} = \frac{\beta\gamma}{\alpha}+\frac{\alpha\gamma}{\beta}+\frac{\beta\alpha}{\gamma}$

$= \frac{\beta^2\gamma^2+\alpha^2\gamma^2+\beta^2\alpha^2}{\alpha\beta\gamma}$.

$$= \frac{5}{(-1)} = -5.$$

vii) $\Sigma \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\gamma^2+\alpha^2}{\gamma\alpha} + \frac{\alpha^2+\beta^2}{\alpha\beta}$

$$= \frac{\alpha^2\beta\gamma(\beta^2+\gamma^2) + \beta^2\alpha\gamma(\gamma^2+\alpha^2) + \gamma^2\beta\alpha(\alpha^2+\beta^2)}{\alpha^2\beta^2\gamma^2}$$

$= -\alpha(-2-\alpha^2) - \beta(-2-\beta^2) - \gamma(-2-\gamma^2).$

$= \alpha(2+\alpha^2) + \beta(2+\beta^2) + \gamma(2+\gamma^2).$

$= 2\alpha + \alpha^3 + 2\beta + \beta^3 + 2\gamma + \gamma^3.$

$= 2(\alpha+\beta+\gamma) + \alpha^3+\beta^3+\gamma^3 + 3(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) - 3(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

$= 2(-2) + (\alpha+\beta+\gamma)^3 - 3(-2-\gamma)(-2-\alpha)(-2-\beta).$

$= -4 + (-2)^3 + (6+3\gamma)(2+\alpha)(2+\beta).$

$= -4 - 8 + (12 + 6\alpha + 6\gamma + 3\gamma\alpha)(2+\beta).$

$= -4 - 8 + 24 + 12\alpha + 12\gamma + 6\gamma\alpha + 12\beta + 6\alpha\beta + 6\gamma\beta + 3\gamma\alpha\beta$

$= -4 - 8 + 24 + 12(\alpha+\beta+\gamma) + 6(\gamma\alpha + \alpha\beta + \beta\gamma) + 3(-1).$

$= -4 - 8 + 24 + 12(-2) + 6(3) - 3.$

$= -4 - 8 + \cancel{24} - \cancel{24} + 18 - 3.$

$= -12 + 18 - 3$

$= 6 - 3.$

$= 3.$

$\alpha\beta\gamma$ (-1) $(\alpha\beta)$

viii) $\Sigma\alpha^3\beta^2 = \alpha^3\beta^2 + \beta^3\gamma^2 + \gamma^3\alpha^2 + \beta^3\alpha^2 + \alpha^3\gamma^2 + \gamma^3\beta^2.$

$= \alpha^2\beta^2(\alpha+\beta) + \beta^2\gamma^2(\beta+\gamma) + \gamma^2\alpha^2(\gamma+\alpha).$

$= \alpha^2\beta^2(-2-\gamma) + \beta^2\gamma^2(-2-\alpha) + \gamma^2\alpha^2(-2-\beta)$

$= -2\alpha^2\beta^2 - \alpha^2\beta^2\gamma - 2\beta^2\gamma^2 - \beta^2\gamma^2\alpha - 2\gamma^2\alpha^2 - \gamma^2\alpha^2\beta$

$= -2(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) + \alpha\beta + \beta\gamma + \gamma\alpha$

DATE
PAGE

$= -2(5) + 3$

$= -10 + 3$

$= -7.$

* — * — * — * — *

Example:- $x^4 + px^3 + qx^2 + rx + s = 0.$

$\Sigma\alpha^2\beta = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + \beta^2\alpha + \gamma^2\beta + \alpha^2\gamma + \delta^2\alpha + \delta^2\beta + \delta^2\gamma$

$+ \alpha^2\delta + \beta^2\delta + \gamma^2\delta$

$= \cancel{\alpha^2\beta^2\gamma^2\delta^2(\beta + \alpha + \delta + \gamma) + \beta^2\gamma^2\alpha^2\delta^2(\gamma + \alpha + \delta + \beta)}$

$\cancel{+ \gamma^2\delta^2\beta^2\alpha^2(\beta + \alpha + \delta + \gamma)}$

$\cancel{= s(p) + s(p) + s(p)}$

$\cancel{= -3sp}$

$= \alpha(\alpha\beta + \alpha\delta + \alpha\gamma) + \beta(\beta\gamma + \beta\delta + \beta\alpha) + \gamma(\gamma\delta + \gamma\alpha + \gamma\beta) + \delta(\delta\beta + \delta\gamma + \delta\alpha)$

$\alpha(\alpha\beta + \alpha\delta + \alpha\gamma + \beta\gamma + \beta\delta + \gamma\delta) = \alpha^2\beta + \alpha^2\delta + \alpha^2\gamma + \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta - \beta\gamma\delta$

$\alpha\Sigma\alpha\beta = \alpha(\beta\alpha + \alpha\delta + \alpha\gamma) + \Sigma\alpha\beta\gamma - \beta\gamma\delta.$

$\alpha\Sigma\alpha\beta - \Sigma\alpha\beta\gamma + \beta\gamma\delta = \alpha(\beta\alpha + \alpha\delta + \alpha\gamma).$ → Put in eq. (1)

$= \alpha\Sigma\alpha\beta - \Sigma\alpha\beta\gamma + \beta\gamma\delta + \beta\Sigma\alpha\beta - \Sigma\alpha\beta\gamma + \alpha\gamma\delta + \gamma\Sigma\alpha\beta - \Sigma\alpha\beta\gamma + \alpha\beta\delta$
$+ \delta\Sigma\alpha\beta - \Sigma\alpha\beta\gamma + \alpha\beta\gamma$

$= \Sigma\alpha\beta(\alpha + \beta + \gamma + \delta) - 4\Sigma\alpha\beta\gamma + \beta\gamma\delta + \alpha\beta\delta + \alpha\beta\delta + \alpha\gamma\delta$

$= \Sigma\alpha\beta(-p).$ $\quad -3\Sigma\alpha\beta\gamma$

DATE
PAGE

$= -pq - 3(-s)$

$= 3r - pq.$

(iii) $\Sigma\alpha^2\beta\gamma = \alpha^2\beta\gamma + \alpha^2\gamma\delta + \alpha^2\delta\beta$
$+ \beta^2\alpha\gamma + \beta^2\gamma\delta + \beta^2\delta\alpha$
$+ \gamma^2\alpha\beta + \gamma^2\beta\delta + \gamma^2\delta\alpha$
$+ \delta^2\alpha\beta + \delta^2\beta\gamma + \delta^2\alpha\gamma.$

$\alpha\beta\gamma(\alpha+\beta+\gamma) + \alpha\gamma\delta(\alpha+\gamma+\delta) + \alpha\delta\beta(\alpha+\delta+\beta) + \beta\gamma\delta(\beta+\gamma+\delta)$
$= \alpha\beta\gamma(-p-\delta) + \alpha\gamma\delta(-p-\beta) + \alpha\delta\beta(-p-\gamma) + \beta\gamma\delta(-p-\alpha).$
$= -p(\alpha\beta\gamma + \alpha\gamma\delta + \alpha\delta\beta + \beta\gamma\delta) - 4\alpha\beta\gamma\delta.$
$= -p(-r) - 4(s)$
$= pr - 4s.$

Ex $x^4 + px + q = 0$, Roots of eqⁿ are $\alpha, \beta, \gamma, \delta$.
Find $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2) = \Pi(1+\alpha^2)$.
we have to make new eqⁿ of roots $\alpha^2, \beta^2, \gamma^2, \delta^2$.
(1) Put $y = x^2$.

$y^2 + px + q = 0.$

$y^2 = -px - q$

$(y^2+q) = -px$

squaring
$(y^2+q)^2 = p^2x^2$
$y^4 + q^2 + 2qy^2 = p^2y$

$\cancel{y^4 + (2q + p^2)y}$ $y^4 + 2qy^2 - p^2y + q^2 = 0.$

we know,

$\alpha^2, \beta^2, \gamma^2, \delta^2$ are the roots of $y^4 + 2y^2q - p^2y + q^2 = 0$

$(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2)$

$= (1+\alpha^2+\beta^2+\alpha^2\beta^2)(1+\gamma^2+\delta^2+\delta^2\gamma^2)$.

$= (1+\gamma^2+\delta^2+\delta^2\gamma^2+\alpha^2+\alpha^2\gamma^2+\alpha^2\delta^2+\alpha^2\delta^2\gamma^2+\beta^2+\beta^2\gamma^2+\beta^2\delta^2+\beta^2\delta^2\gamma^2+$
$\alpha^2\beta^2+\alpha^2\beta^2\gamma^2+\alpha^2\beta^2\delta^2+\alpha^2\beta^2\delta^2\gamma^2$.

$= 1+\overbrace{(\gamma^2+\delta^2+\alpha^2+\beta^2)}^{0}+(\delta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2+\alpha^2\beta^2)$
$+(\alpha^2\delta^2\gamma^2+\beta^2\delta^2\gamma^2+\alpha^2\beta^2\gamma^2+\alpha^2\beta^2\delta^2)+\alpha^2\beta^2\delta^2\gamma^2$.

$= 1+2q+p^2+q^2$.

$= p^2+q^2+2q+1$

Ex (1) Consider the eq^n.

$x^3+px^2+qx+r=0$.

find i) $\Sigma\alpha^3 = \alpha^3+\beta^3+\gamma^3$.

Put $y=x^3$. [we have to form new eq. whose roots are the cube of the roots of given eq^n.]

$y+px^2+qx+r=0$.

$y+r = -x(px+q)$. — (1)

squaring eq. (1)

$(y+r)^3 = -x^3(px+q)^3$

$y^3+r^3+3r^2y+3y^2r = -y[p^3y+q^3+3q^2px+3p^2qx^2]$

$$y^3+3ry^2+3r^2y+r^3=-p^3y^2-q^3y-3pqy\cdot r(q+px)$$

$$y^3+3ry^2+3r^2y+r^3=-p^3y^2-q^3y+3pqy(y+r) \quad \text{using eq. (1)}$$

$$y^3+3ry^2+p^3y^2+3r^2y+q^3y+r^3=+3pqy^2+3pqry$$

$$y^3+(p^3+3r-3pq)y^2+(3r^2+q^3-3pqr)y+r^3=0$$

$$\therefore \Sigma\alpha^3=-(p^3+3r-3pq)$$
$$=-p^3-3r+3pq$$

iii) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

$=(\beta\gamma+\beta\alpha+\gamma^2+\gamma\alpha)(\alpha+\beta)$

$=\alpha\beta\gamma+\alpha^2\beta+\gamma^2\alpha+\alpha^2\gamma+\beta^2\gamma+\beta^2\alpha+\gamma^2\beta+\alpha\beta\gamma$

$=\alpha\beta(\alpha+\beta)+\gamma\alpha(\gamma+\alpha)+\beta\gamma(\beta+\gamma) \quad -2r$

$=\alpha\beta(-p-\gamma)+\gamma\alpha(-p-\beta)+\beta\gamma(-p-\alpha)-2r$

$=-p(\alpha\beta+\gamma\alpha+\beta\gamma)-\alpha\beta\gamma-\alpha\beta\gamma-\alpha\beta\gamma-2r$

$\cancel{=-pq+3r-2r}$

$=-pq-3(-r)-2r$

$=-pq+3r-2r$

$=r-pq$

(2) For the biquadratic eq^n:

$x^4+px^3+qx^2+rx+s=0$. Roots are $\alpha,\beta,\gamma,\delta$.

i) $\Sigma\alpha^2=\alpha^2+\beta^2+\gamma^2+\delta^2$

Put $x^2=y$ [we have to form new eq. whose roots are the square of the roots of given eq^n]

$y^2+pxy+qy+rx+s=0$

$y(px+q) = (-1)(y^2+rx+s)$

squaring

$y^2[p^2x^2+q^2+2pqx] = [y^4+r^2x^2+s^2+2y^2rx+2rsx+2sy^2]$

$y^2[p^2y+q^2$

$x(py+r) = (-1)(y^2+qy+s).$

squaring

$x^2(p^2y^2+r^2+2rpy] = [y^4+q^2y^2+s^2+2y^3q+2qsy+2sy^2]$

$p^2y^3+r^2y+2rpy^2 = y^4+q^2y^2+s^2+2y^3q+2qsy+2sy^2.$

$0 = y^4+(2q-p^2)y^3+(q^2+2s-2rp)y^2+(2qs-r^2)y+s^2 = 0$

$\Sigma\alpha^2 = -(2q-p^2)$
$= p^2-2q.$

ii) $\Sigma\alpha^2\beta^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\delta^2+\delta^2\alpha^2.$
$= q^2+2s-2rp.$

iii) $\Sigma\alpha^4 = \alpha^4+\beta^4+\gamma^4+\delta^4.$
$= (\alpha^2)^2+(\beta^2)^2+(\gamma^2)^2+(\delta^2)^2$
$= (\alpha^2)^2+(\beta^2)^2+(\gamma^2)^2+(\delta^2)^2+2\alpha^2\beta^2+2\beta^2\gamma^2+2\gamma^2\delta^2+2\delta^2\alpha^2$
$-2\alpha^2\beta^2-2\beta^2\gamma^2-2\gamma^2\delta^2-2\delta^2\alpha^2$
$= (\alpha^2+\beta^2+\gamma^2+\delta^2)^2-2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\delta^2+\delta^2\alpha^2).$
$= (p^2-2q)^2-2(q^2+2s-2rp)$
$= p^4+4q^2-4qp^2-2q^2-4s+4rp.$
$= p^4-4p^2q+2q^2-4s+4rp$

iv) $\Sigma(\alpha-\beta)^4 = (\alpha-\beta)^4+(\beta-\gamma)^4+(\gamma-\delta)^4+(\delta-\alpha)^4+(\alpha-\gamma)^4+(\beta-\delta)^4$.

$=(\alpha^2+\beta^2-2\alpha\beta)^2+(\beta^2+\gamma^2-2\gamma\beta)^2+(\gamma^2+\delta^2-2\gamma\delta)^2+(\delta^2+\alpha^2-2\delta\alpha)+(\alpha^2+\gamma^2-2\alpha\gamma)^2$
$+(\beta^2+\delta^2-2\beta\delta)^2$

$=\alpha^4+\beta^4+4\alpha^2\beta^2+2\alpha^2\beta^2-4\alpha\beta^3-4\alpha^3\beta$
$+\beta^4+\gamma^4+4\gamma^2\beta^2+2\gamma^2\beta^2-4\gamma\beta^3-4\gamma^3\beta$
$+\gamma^4+\delta^4+4\delta^2\gamma^2+2\delta^2\gamma^2-4\delta\gamma^3-4\delta^3\gamma$.
$+\delta^4+\alpha^4+4\delta^2\alpha^2+2\delta^2\alpha^2-4\delta\alpha^3-4\alpha^3\delta$
$+\alpha^4+\gamma^4+4\alpha^2\gamma^2+2\alpha^2\gamma^2-4\alpha\gamma^3-4\alpha^3\gamma$.
$+\beta^4+\delta^4+4\beta^2\delta^2+2\beta^2\delta^2-4\beta\delta^3-4\beta^3\delta$

$=3(\alpha^4+\beta^4+\delta^4+\gamma^4)+6(\alpha^2\beta^2+\gamma^2\beta^2+\gamma^2\delta^2+\delta^2\alpha^2+\alpha^2\gamma^2+\beta^2\delta^2)$ ~~$-4\alpha^3(\beta+\gamma+\delta)$~~
~~$-4\beta^3(\alpha+\gamma+\delta)-4\gamma^3(\alpha+\beta+\delta)$~~
$-4\alpha\beta(\alpha^2+\beta^2)-4\beta\gamma(\beta^2+\gamma^2)-4\delta\gamma(\gamma^2+\delta^2)-4\delta\alpha(\delta^2+\alpha^2)-4\alpha\gamma(\alpha^2+\gamma^2)$
$-4\beta\delta(\delta^2+\beta^2)$.

$=3\Sigma\alpha^4+6\Sigma\alpha^2\beta^2-4\alpha\beta(\alpha^2+\beta^2)-4\beta\gamma(\beta^2+\gamma^2)-4\delta\gamma(\gamma^2+\delta^2)-4\delta\alpha(\delta^2+\alpha^2)-4\alpha\gamma(\alpha^2+\gamma^2)-4\beta\delta$
$(\beta^2+\delta^2)$ — (1)

$4\alpha\beta(\alpha^2+\beta^2+\gamma^2+\delta^2)=4\alpha^3\beta+4\alpha\beta^3+4\alpha\beta\gamma^2+4\alpha\beta\delta^2$
$4\alpha\beta\Sigma\alpha^2=4\alpha^3\beta+4\alpha\beta^3+4\alpha\beta\gamma^2+4\alpha\beta\delta^2$
$4\alpha\beta\Sigma\alpha^2-4\alpha\beta\gamma^2-4\alpha\beta\delta^2=4\alpha^3\beta+4\alpha\beta^3$

Put in eq. (1)

$3\Sigma\alpha^4+6\Sigma\alpha^2\beta^2-4\Sigma\alpha^2(\alpha\beta+\beta\gamma+\delta\gamma+\delta\alpha+\alpha\gamma+\beta\delta)+4(\alpha\beta\gamma^2+\alpha\beta\delta^2+\beta\gamma\delta^2+\beta\gamma\alpha^2$
$+\delta\gamma\beta^2+\gamma\delta\alpha^2+\delta\alpha\beta^2+\delta\alpha\gamma^2$
$+\alpha\gamma\beta^2+\alpha\gamma\delta^2+\delta\beta\gamma^2+$
$\delta\beta\alpha^2)$

$3\Sigma\alpha^4+6\Sigma\alpha^2\beta^2-4\Sigma\alpha^2\Sigma\alpha\beta+4\Sigma\alpha^2\beta\gamma$.

~~$=3(p^2-2q)+6(q^2+2s-2rp)-4(p^2-2q)(q)+4(pr-4s)$~~
$=3(p^4-4p^2q+2q^2-4s+4rp)+6(q^2+2s-2rp)-4(p^2-2q)(q)+4pr-16s$
$=3p^4-12p^2q+6q^2-12s+12rp+6q^2+12s-12rp-4p^2q-8q^2+4pr-16s$
$=3p^4-16p^2q+20q^2+4pr-16s$.

Scanned with CamScanner

For $a_0x^3+3a_1x^2+3a_2x+a_3=0$.
Find value of
$$(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)$$

$$(2\alpha-\beta-\gamma) = 2\alpha+2\beta+2\gamma-\beta-\gamma-2\beta-2\gamma$$
$$= 2\alpha+2\beta+2\gamma-3\beta-3\gamma$$
$$= 2(\alpha+\beta+\gamma)-3(\beta+\gamma)$$
$$= -\frac{6a_1}{a_0}-3\left(\frac{-3a_1}{a_0}-\alpha\right)$$
$$= -\frac{6a_1}{a_0}+\frac{9a_1}{a_0}+3\alpha$$
$$= \frac{3a_1}{a_0}+3\alpha$$

$2\alpha+2\beta+2\gamma-\beta-\gamma$

Similarly for $(2\beta-\gamma-\alpha)=\frac{3a_1}{a_0}+3\beta$
$$(2\gamma-\alpha-\beta)=\frac{3a_1}{a_0}+3\gamma$$

$$27\left(\frac{a_1}{a_0}+\alpha\right)\left(\frac{a_1}{a_0}+\beta\right)\left(\frac{a_1}{a_0}+\gamma\right)$$

$$=27\left[\left(\frac{a_1}{a_0}\right)^2+\frac{a_1}{a_0}\beta+\frac{a_1}{a_0}\alpha+\alpha\beta\right]\left(\frac{a_1}{a_0}+\gamma\right)$$

$$=27\left[\left(\frac{a_1}{a_0}\right)^3+\left(\frac{a_1}{a_0}\right)^2\beta+\left(\frac{a_1}{a_0}\right)^2\alpha+\frac{a_1}{a_0}\alpha\beta+\left(\frac{a_1}{a_0}\right)^2\gamma+\frac{a_1}{a_0}\gamma\beta+\frac{a_1}{a_0}\gamma\alpha+\alpha\beta\right]$$

$$=27\left[\left(\frac{a_1}{a_0}\right)^3+\left(\frac{a_1}{a_0}\right)^2(\beta+\alpha+\gamma)+\frac{a_1}{a_0}(\alpha\beta+\beta\gamma+\gamma\alpha)+\alpha\beta\gamma\right]$$

for

(4) $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$

find

i) $(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2+(\alpha-\beta)^2(\gamma-\delta)^2$

$$=27\left[\left(\frac{a_1}{a_0}\right)^3+\left(\frac{a_1}{a_0}\right)^2\left(-\frac{3a_1}{a_0}\right)+\frac{a_1}{a_0}\left(\frac{3a_2}{a_0}\right)-\frac{a_3}{a_0}\right]$$

$$=\frac{27}{a_0^3}\left[a_1^3-3a_1a_1^2+3a_0a_2a_1-a_3a_0^2\right]$$

$$=\frac{-27}{a_0^3}\left[3a_1a_1^2+a_3a_0^2-3a_0a_2a_1-a_1^3\right]$$

$$=\frac{-27}{a_0^3}\left[2a_1^3-3a_0a_1a_2+a_3a_0^2\right]$$

(4) For $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$.

Find.

i) $(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2+(\alpha-\beta)^2(\gamma-\delta)^2=$

$(\beta-\gamma)^2(\alpha-\delta)^2=(\beta^2+\gamma^2-2\beta\gamma)(\alpha^2+\delta^2-2\alpha\delta)$

$=\beta^2\alpha^2+\beta^2\delta^2-2\beta^2\alpha\delta+\gamma^2\alpha^2+\gamma^2\delta^2-2\gamma^2\alpha\delta+4\beta\gamma\alpha\delta-2\alpha^2\beta\gamma-2\delta^2\beta\gamma$.

$(\gamma-\alpha)^2(\beta-\delta)^2=\gamma^2\beta^2+\gamma^2\delta^2-2\gamma^2\beta\delta+\alpha^2\beta^2+\alpha^2\delta^2-2\alpha^2\beta\delta-2\beta^2\alpha\gamma-2\delta^2\alpha\gamma+4\beta\gamma\alpha\delta$

$(\alpha-\beta)^2(\gamma-\delta)^2=\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2-2\alpha^2\gamma\delta-2\beta^2\gamma\delta-2\gamma^2\alpha\beta-2\delta^2\alpha\beta+4\beta\gamma\alpha\delta$.

DATE
PAGE

$(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2+(\alpha-\beta)^2(\gamma-\delta)^2$

$= \beta^2\alpha^2+\beta^2\delta^2+\alpha^2\gamma^2+\delta^2\gamma^2-2\gamma^2\alpha\delta-2\alpha^2\beta\gamma-2\delta^2\beta\gamma-2\beta^2\alpha\delta+4\beta\gamma\delta\alpha$

$\gamma^2\beta^2+\gamma^2\delta^2+\alpha^2\delta^2+\alpha^2\beta^2-2\gamma^2\beta\delta-2\alpha^2\beta\delta-2\delta^2\alpha\gamma-2\beta^2\alpha\gamma+4\beta\gamma\delta\alpha$

$\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2-2\gamma^2\alpha\beta-2\alpha^2\gamma\delta-2\delta^2\alpha\beta-2\beta^2\gamma\delta+4\beta\gamma\delta\alpha$

$= 2(\beta^2\alpha^2+\beta^2\delta^2+\alpha^2\gamma^2+\delta^2\gamma^2+\gamma^2\beta^2+\alpha^2\delta^2) - 2(\gamma^2\alpha\delta+\gamma^2\beta\delta+\gamma^2\alpha\beta+$
$\beta^2\alpha\delta+\beta^2\alpha\gamma+\beta^2\gamma\delta+$
$\alpha^2\beta\gamma+\alpha^2\beta\delta+\alpha^2\gamma\delta+$
$\delta^2\beta\gamma+\delta^2\alpha\gamma+\delta^2\alpha\beta)+12\alpha\beta\delta\gamma$

$= 2\Sigma\alpha^2\beta^2 - 2\Sigma\alpha^2\beta\gamma + 12\alpha\beta\delta\gamma$ — (1)

~~We~~ ~~Note~~ We have form eq. whose roots are square of the roots of given eq.

$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$

Put $y=x^2$

$a_0y^2+4a_1xy+6a_2y+4a_3x+a_4=0$

$(a_0y^2+6a_2y+a_4) = -4x(a_1y+a_3)$

squaring b.s.

$(a_0y^2)^2+(6a_2y)^2+(a_4)^2+2(a_0y^2)(6a_2y)+2(6a_2y)(a_4)+2(a_4)(a_0y^2)$
$= 16x^2[(a_1y)^2+(a_3)^2+2a_1a_3y]$

$a_0^2y^4+36a_2^2y^2+a_4^2+12a_0a_2y^3+12a_2a_4y+2a_0a_4y^2 = 16y[a_1^2y^2+a_3^2+2a_1a_3y]$

$a_0^2y^4+(36a_2^2+2a_0a_4-32a_1a_3)y^2+(12a_2a_4-16a_3^2)y+a_4^2+(12a_0a_2-16a_1^2)y^3=0$

$a_0^2y^4+(12a_0a_2-16a_1^2)y^3+(36a_2^2+2a_0a_4-32a_1a_3)y^2+(12a_2a_4-16a_3^2)y+a_4^2=0$

$\therefore \Sigma\alpha^2\beta^2 = \dfrac{36a_2^2 + 2a_0a_4 - 32a_1a_3}{a_0^2}$

Now, $\Sigma\alpha^2\beta\gamma = \alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha^2\gamma\delta +$
$\beta^2\alpha\delta + \beta^2\alpha\gamma + \beta^2\gamma\delta +$
$\gamma^2\beta\alpha + \gamma^2\alpha\delta + \gamma^2\beta\delta +$
$\delta^2\beta\gamma + \delta^2\alpha\gamma + \delta^2\alpha\beta$

$\alpha\beta\gamma(\alpha+\beta+\gamma) + \alpha\beta\delta(\alpha+\beta+\delta) + \alpha\gamma\delta(\alpha+\gamma+\delta) + \beta\gamma\delta(\beta+\gamma+\delta)$

$= \alpha\beta\gamma\left(-\dfrac{4a_1}{a_0} - \delta\right) + \alpha\beta\delta\left(-\dfrac{4a_1}{a_0} - \gamma\right) + \alpha\gamma\delta\left(-\dfrac{4a_1}{a_0} - \beta\right) + \beta\gamma\delta\left(-\dfrac{4a_1}{a_0} - \alpha\right)$

$= -\dfrac{4a_1}{a_0}(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) - 4\alpha\beta\gamma\delta$

$= -\dfrac{4a_1}{a_0}\left(\dfrac{-4a_3}{a_0}\right) - 4\left(\dfrac{a_4}{a_0}\right)$

$= \dfrac{16a_1a_3}{a_0^2} - \dfrac{4a_4}{a_0}$

Put $\Sigma\alpha^2\beta^2$ & $\Sigma\alpha^2\beta\gamma$.

$\dfrac{2(36a_2^2 + 2a_0a_4 - 32a_1a_3)}{a_0^2} - 2\left(\dfrac{16a_1a_3}{a_0^2} - \dfrac{4a_4}{a_0}\right) + 12\dfrac{a_4}{a_0}$

$\dfrac{72a_2^2}{a_0^2} + \dfrac{4a_0a_4}{a_0^2} - \dfrac{64a_1a_3}{a_0^2} - \dfrac{32a_1a_3}{a_0^2} + \dfrac{8a_4}{a_0} + \dfrac{12a_4}{a_0}$

$-16a_1^2)y^3 = 0.$

$= 0$

$\dfrac{72a_2^2}{a_0^2} + \dfrac{4a_0a_4}{a_0^2} - \dfrac{96a_1a_3}{a_0^2} + \dfrac{20a_0a_4}{a_0^2}$

$= \dfrac{72a_2^2}{a_0^2} + \dfrac{24a_0a_4}{a_0^2} - \dfrac{96a_1a_3}{a_0^2}$

DATE
PAGE

$$\frac{24}{a_0^2}\left[3a_2^2 + a_0a_4 - 4a_1a_3\right]$$

(ii) $\Sigma(\alpha-\beta)^2 = (\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\delta)^2 + (\delta-\alpha)^2 + (\beta-\delta)^2 + (\gamma-\alpha)^2$

$= \alpha^2 + \beta^2 - 2\alpha\beta + \beta^2 + \gamma^2 - 2\beta\gamma + \gamma^2 + \delta^2 - 2\gamma\delta + \delta^2 + \alpha^2 - 2\alpha\delta + \beta^2 + \delta^2 - 2\beta\delta + \gamma^2 + \alpha^2 - 2\alpha\gamma$

$= 3\alpha^2 + 3\beta^2 + 3\gamma^2 + 3\delta^2 - 2\alpha\beta - 2\beta\gamma - 2\gamma\delta - 2\alpha\delta - 2\beta\delta - 2\alpha\gamma$

$= 3(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) - 2(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \beta\delta + \alpha\gamma)$

$= \frac{3(-12a_0a_2 + 16a_1^2)}{a_0^2} - 2\left(\frac{6a_2}{a_0}\right)$

$= -\frac{36a_0a_2}{a_0^2} + \frac{48a_1^2}{a_0^2} - \frac{12a_2}{a_0} \times \frac{a_0}{a_0}$

$= \frac{48a_1^2}{a_0^2} - \frac{48a_2a_0}{a_0^2}$

$= \frac{48}{a_0^2}(a_1^2 - a_2a_0)$

Def :

Exa

Scanned with CamScanner

Reciprocal Eq^n :

Def : Let $f(x)=0$ be an eq^n, whose roots are $\alpha_1, \alpha_2, \ldots \alpha_n$. If $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$ are also the roots of $f(x)$, then $f(x)$ is called reciprocal eq^n.

Example $\Rightarrow$ $2x^2-5x+2$
$= 2x^2-4x-x+2$
$= 2x(x-2)-(x-2)$
$= (2x-1)(x-2)$
$x = \frac{1}{2}, 2$

$(2x^2-5x+2)(x-1)$
$x = \frac{1}{2}, 2, 1$

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$ be a reciprocal equation.

Let $y = \frac{1}{x}$

$\Rightarrow x = \frac{1}{y}$

$\frac{a_n}{a_0} = \frac{a_0}{a_n}$

$g(y) = \frac{a_0}{y^n} + \frac{a_1}{y^{n-1}} + \frac{a_2}{y^{n-2}} + \ldots + a_n$

$g(y) = a_0 + a_1y + a_2y^2 + \ldots + a_ny^n = 0$

$g(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

As $f(x)$ is reciprocal eq^n,
$f(x)$ & $g(x)$ has same roots

$\Rightarrow \frac{a_0}{a_n} = \frac{a_1}{a_{n-1}} = \frac{a_2}{a_{n-2}} = \ldots = \frac{a_n}{a_0} = k$

$\frac{1}{\frac{a_n}{a_0}} = \frac{a_n}{a_0} = k.$

$\Rightarrow \frac{1}{k} = k$

$\Rightarrow k^2 = 1 \Rightarrow k = \pm 1.$

If $k = 1$. $a_0 = a_n ; a_1 = a_{n-1} , \ldots , a_n = a_0.$

If $k = -1$ $a_0 = -a_n , a_1 = -a_{n-1} , \ldots , a_n = -a_0.$

~~Other~~ The Reciprocal equations for which $k = 1$ are called reciprocal eq^n. of 1st type.

~~The~~ and for which $k = -1$, are called reciprocal eq^n. of 2nd type.

* If a reciprocal eq^n. is of 1st type and of even degree, then it is called standard reciprocal Equation.

* If a reciprocal eq^n. is of 1st type & odd degree, then $x = -1$ is always its root.

$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n.$

$n \to$ odd. $\therefore n = 2m+1.$

Product of roots $= (-1)^n \frac{a_n}{a_0} = -\frac{a_n}{a_0}$

$= -1.$ $[\because a_n = a_0]$

$\left(\alpha_1 \cdot \frac{1}{\alpha_1}\right)\left(\alpha_2 \cdot \frac{1}{\alpha_2}\right)\left(\alpha_3 \cdot \frac{1}{\alpha_3}\right) \ldots \left(\alpha_m \cdot \frac{1}{\alpha_m}\right) \alpha_{m+1} = -1$

$\alpha_{m+1} = -1.$

DATE
PAGE

* If a reciprocal eqⁿ is of 2nd type & odd degree, then $x = 1$ is always it's roots.

Solution of Cubic & Biquadratic Eqⁿ :-

① Cardon's Method (for degree 3 eqⁿs) :-
Used to solve equations of the form

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0. \quad \text{— ①}$$

Reduce eq. ① to the form $z = a_0x + a_1$

$$\frac{z - a_1}{a_0} = x.$$

$\alpha + \beta + \gamma = -3a_1$

$-3a_1 + 3a_1$

$\alpha + a_1 + \beta + a_1 + \gamma - a_1 =$

$\alpha + \beta + \gamma - (3a_1)$

$-3a - 3$

$-6a_1$

$\alpha + \beta + \gamma = \frac{-3a_1}{a_0}$

$$z^3 + 3Hz + G = 0. \quad \text{— ②}$$

Put $z = u + v$.
Cubing both side

$$z^3 = u^3 + v^3 + 3u^2v + 3uv^2$$

$$= u^3 + v^3 + 3uv(u+v)$$

$$z^3 = u^3 + v^3 + 3uvz.$$

$$z^3 - 3uvz - (u^3 + v^3) = 0 \quad \text{— ③}$$

Compare ② & ③, we get.

$$H = -uv$$

$$G = -(u^3 + v^3)$$

$$-H = uv$$

$$-H^3 = u^3v^3.$$

$\Rightarrow$ u^3 & v^3 are roots of Eqⁿ

$$x^2 + Gx - H^3 = 0.$$

$$v^3, u^3 = \frac{-G \pm \sqrt{G^2 + 4H^3}}{2}$$

Scanned with CamScanner

Find the value of u & v from here & then $u+v=z$ gives the roots of eq (1).

Example:- $28x^3 - 9x^2 + 1 = 0$.

$z^3+3Hz+G=0$.

$z=u+v$.

Cubing.

$z^3=u^3+v^3+3uvz$.

$-uv=H$

$-(u^3+v^3)=G$.

Put $x=\frac{1}{y}$

$\frac{28}{y^3} - \frac{9}{y^2} + 1 = 0$

$y^3 - 9y + 28 = 0$. —(2)

Put $y=u+v$

Cubing

$y^3 = u^3+v^3+3uvy$.

$y^3 - 3uvy - (u^3+v^3) = 0$. —(3)

$H = -3uv$

$G = -(u^3+v^3)$

Compare eq (2) & (3)

$-9 = -3uv$

$28 = -(u^3+v^3)$

$uv = 3$ $\Rightarrow u^3v^3 = 27$.

$u^3+v^3 = -28$.

u^3, v^3 are roots of eqn. $x^2+28x+27=0$.

$x^2+27x+x+27=0$

$x(x+27)+(x+27)=0$

$(x+1)(x+27)=0$.

$x=-27, -1$.

$u^3=-27$, $v^3=-1$

$u=-3$ is one of the value

$v=-1$ is one of the value

$u^3 = -27.$

$u = -1 \cdot 27$

$= (-1)^{1/3} (27)^{1/3}.$

$= 3 (-1)^{1/3}.$

$-1 = r\cos\theta + i r\sin\theta.$

$r\cos\theta = -1$ — ①

$r\sin\theta = 0$ — ②

Squaring & adding

$r^2 = 1$

$\Rightarrow r = 1.$

$\theta = \pi.$

$(-1)^{1/3} = (\text{cis}\,\pi)^{1/3} = \text{cis}\left(\frac{\pi + 2n\pi}{3}\right)$; $n = 0, 1, 2.$

$y = u + v = -3 - 1 = -4.$

$y^3 - 9y + 28 = 0.$

$$\begin{array}{r|rrr|r} & y^3 & y^2 & y & c \\ -4 & 1 & 0 & -9 & 28 \\ & & -4 & 16 & -28 \\ \hline & 1 & -4 & 7 & \boxed{0} \end{array}$$

$y^2 - 4y + 7 = 0.$

$y = \frac{4 \pm \sqrt{16 - 28}}{2}$

$= \frac{4 \pm \sqrt{-12}}{2} \Rightarrow \frac{4 \pm 2\sqrt{3}\,i}{2} = 2 \pm \sqrt{3}\,i$

DATE
PAGE

We get $y = -4, 2 \pm \sqrt{3}i$

$x = -\frac{1}{4}, \frac{1}{2 \pm \sqrt{3}i}$

$\frac{1}{2+\sqrt{3}i} \times \frac{2-\sqrt{3}i}{2-\sqrt{3}i} \Rightarrow \frac{2-\sqrt{3}i}{4+3} = \frac{2-\sqrt{3}i}{7}$

$\frac{1}{2-\sqrt{3}i} \times \frac{2+\sqrt{3}i}{2+\sqrt{3}i} = \frac{2+\sqrt{3}i}{4+3} = \frac{2+\sqrt{3}i}{7}.$

Cardon's Method :-

$2x^3 - 7x^2 + 8x - 3 = 0$ — (1)

$y = 6x$

$\Rightarrow x = \frac{y}{6}.$

$2\left(\frac{y}{6}\right)^3 - 7\left(\frac{y}{6}\right)^2 + 8\left(\frac{y}{6}\right) - 3 = 0.$

$2y^3 - 42y^2 + 288y - 648 = 0.$

$2(y^3 - 21y^2 + 144y - 324) = 0$

$y^3 - 21y^2 + 144y - 324 = 0.$ — (2)

$z^3 + 3Hz + G = 0$

$z = u + v$

$H = -uv$

$-H^3 = u^3v^3$

$G^2 = (u^3 + v^3)$

$\alpha + \beta + \alpha = \frac{7}{2}$

$\alpha + \beta + \alpha - \frac{7}{6} = 0$

$(\alpha - a) + (\beta - a) + (\gamma - a) = 0$

$\alpha + \beta + \gamma - 3a = 0.$

Compare

$3a = \frac{7}{2}$

$a = \frac{7}{6}.$

DATE
PAGE

	y^3	y^2	y	C
7	1	-21	144	-324
		7	-98	322
7	1	-14	46	$\boxed{-2}$
		7	-49	
7	1	-7	$\boxed{-3}$	
		7		
	1	$\boxed{0}$		

$z^3 - 3z - 2 = 0$ — (3)

$z = u + v$

$z^3 = u^3 + v^3 + 3uvz$

$z^3 - 3uvz - (u^3+v^3) = 0$ — (4)

Compare

$3uv = 3$

$uv = 1$

$u^3v^3 = 1$

$u^3 + v^3 = 2$

$x^2 - 2x + 1 = 0$

$x^2 - x - x + 1 = 0$

$x(x-1) - (x-1) = 0$

$(x-1)(x-1) = 0$

$x = 1, 1$

$u^3 = 1$

$v^3 = 1$

$u = 1$ is one of the value

$v = 1$ is one of the value

$\therefore z = u + v$

$\boxed{z = 2}$

For remaining roots of eq. (3) we divide (3) by $z-2$

$$\begin{array}{c|cccc} & z^3 & z^2 & z & c \\ 2 & 1 & 0 & -3 & -2 \\ & & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$x = 1z$

$z^2 - \frac{3}{4}z + \frac{1}{2} = 0$

$z^2 - \frac{3}{4}z - \frac{1}{4} = 0$

$z^2 + 2z + 1 = 0$

$z^2 + z + z + 1 = 0$

$z(z+1) + (z+1) = 0$

$(z+1)(z+1) = 0$

$(z+1)^2 = 0.$

$z = -1, -1.$

Roots of z are $2, -1, -1.$

Roots of y are $2+7, -1+7, -1+7.$

$9, 6, 6.$

Roots of x are $\frac{9}{6}, \frac{6}{6}, \frac{6}{6}$

$= \frac{3}{2}, 1, 1.$

DATE
PAGE

Trigonometric Method.

$z^3 + 3Hz + G = 0$ After transformation.

Check $G^2 + 4H^3 < 0$, Use trignometric method.

≥ 0, Use Cardon Method.

Example $x^3 - 3x + 1 = 0$. — (1) $z^3 + 3Hz + G = 0$.

$G = 1$, $H = -1$.

$G^2 + 4H^3 = 1 + 4(-1) = -3 < 0$.

So, we will use trignometric method.

Put $x = tz$.

$t^3z^3 - 3tz + 1 = 0$ — (*)

$z^3 - \frac{3}{t^2}z + \frac{1}{t^3} = 0$. — (2)

C.T.M

Compare with eq. $z^3 - \frac{3}{4}z - \frac{1}{4}\cos 3\theta = 0$

$\frac{-3}{4} = \frac{-3}{t^2}$ $\quad \frac{1}{t^3} = \frac{-1}{4}\cos 3\theta$. $\quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Put $\cos\theta = z$.

$t^2 = 4$, $\quad \cos 3\theta = 4z^3 - 3z$

$t = \pm 2$ $\quad \frac{1}{8} = \frac{-1}{4}\cos 3\theta$ $\quad 4z^3 - 3z - \cos 3\theta = 0$.

$z^3 - \frac{3}{4}z - \frac{1}{4}\cos 3\theta = 0$

is always +ve

$\therefore t = 2$ $\quad \frac{1}{2} = -\cos 3\theta$.

$-\frac{1}{2} = \cos 3\theta$

DATE
PAGE

$3\theta = \cancel{\frac{\pi}{3}}, \frac{2\pi}{3}$

$\theta = \cancel{\frac{\pi}{9}}, \frac{2\pi}{9}$

Roots of Eq^n. (2) are $\cos\theta, \cos\left(\theta + \frac{2\pi}{3}\right), \cos\left(\theta + \frac{4\pi}{3}\right)$

If $\theta = \frac{2\pi}{9}$ $\quad \cos\frac{2\pi}{9}, \cos\frac{8\pi}{9}, \cos\frac{14\pi}{9}$

$\frac{2\pi}{9}, \frac{2\pi}{3}$ $\boxed{\text{If } \cancel{\theta = \frac{\pi}{9}} \quad \cancel{\cos\frac{\pi}{9}}, \cancel{\cos\frac{7\pi}{9}}, \cancel{\cos\frac{13\pi}{9}}}$

$\frac{2\pi}{3}$

$\frac{\pi}{9} + \frac{2\pi}{3}$

$\frac{\pi + 6\pi}{9}$

we Put $x = tz$

we know t & z.

$x = \cancel{\pm} 2\cos\frac{2\pi}{9}$

$x = 2\cos\frac{8\pi}{9}$, $x = 2\cos\frac{14\pi}{9}$

Scanned with CamScanner

DATE
PAGE

$x^3 - 12x + 8 = 0$

$G = 8$

$H = -4 \rightarrow -64$

$G^2 + 4H^3 < 0$

$64 - 4(64) < 0$

$- \quad < 0$

$\sin 30°$	$\cos 60°$
$30°$	$60°$
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

Put $x = tz$.

$t^3 z^3 - 12tz + 8 = 0$.

$z^3 - \frac{12}{t^2} z + \frac{8}{t^3} = 0$. — (2)

Compare $z^3 - \frac{3}{4} z - \frac{1}{4} \cos 3\theta$

$\frac{-3}{4} = \frac{-12}{t^2}$

$\frac{1}{4} = \frac{4}{t^2}$

$t^2 = 16$

$\boxed{t = 4}$

$-\frac{1}{4} \cos 3\theta = \frac{8}{t^3}$

$= \frac{8}{64}$

$-\frac{1}{4} \cos 3\theta = \frac{1}{8}$

$\cos 3\theta = -\frac{1}{2}$

$3\theta = \frac{2\pi}{3}$

$\theta = \frac{2\pi}{9}$

Roots are eq. (2) ie $\cos \theta$, $\cos\left(\theta + \frac{2\pi}{3}\right)$, $\cos\left(\theta + \frac{4\pi}{3}\right)$

DATE
PAGE

Roots of eq. ① is $\cos\frac{2\pi}{9}$, $\cos\frac{8\pi}{9}$, $\cos\left(\frac{14\pi}{9}\right)$

Roots of eq. ① is $4\cos\frac{2\pi}{9}$, $4\cos\frac{8\pi}{9}$, $4\cos\frac{14\pi}{9}$

Hw Questions

① $x^3 - 15x^2 - 33x + 847 = 0$

② $x^3 - 12x - 8\sqrt{2} = 0.$

③ $2x^3 + 3x^2 + 3x + 1 = 0.$

② $x^3 - 12x - 8\sqrt{2} = 0.$ —① $z^3 + 3Hz + G = 0.$

Verify by $G^2 + 4H^3 < 0$ if this is true then use trigonometric method

or

$G^2 + 4H^3 \geq 0$ if this is true then use Cardon meth

$G = -8\sqrt{2}$, $G^2 = 64 \times 2.$

$3H = -12$, $H^3 = -64$

$H = -4$

Now, $G^2 + 4H^3 = 64(2) + 4(-64)$

$= 64(2-4)$

$= 64(-2) < 0$

∴ We have to use ~~cardan~~ trigonometric method

Put $x = tz$

	$30°$	$60°$
Sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

$$t^3z^3 - 12tz - 8\sqrt{2} = 0$$

Now, $\frac{t^3z^3}{t^3} - \frac{12zt}{t^3} - \frac{8\sqrt{2}}{t^3} = 0$

$$z^3 - \frac{12z}{t^2} - \frac{8\sqrt{2}}{t^3} = 0 \quad \text{— (2)}$$

Compare eq. (2) with $z^3 - \frac{3}{4}z - \frac{1}{4}\cos 3\theta$.

$$-\frac{12}{t^2} = -\frac{3}{4}$$

$$\frac{12}{3} = \frac{t^2}{4}$$

$$4 = \frac{t^2}{4}$$

$$16 = t^2$$

$$\boxed{4 = t}$$

$$-\frac{1}{4}\cos 3\theta = -\frac{8\sqrt{2}}{t^3}$$

$$\frac{1}{4}\cos 3\theta = \frac{8\sqrt{2}}{64_8}$$

$$\cos 3\theta = \frac{\cancel{4}\sqrt{2}}{8_2}$$

$$\cos 3\theta = \frac{1}{\sqrt{2}}$$

$\frac{\pi}{4}$

$$3\theta = 45°$$

$$\theta = \frac{45}{3}$$

$\theta = 15°$ i.e. $\theta = \frac{\pi}{12}$.

Roots of eq. (2) is $\cos\frac{\pi}{12}$, $\cos\left(\frac{\pi}{12} + \frac{2\pi}{3}\right)$, $\cos\left(\frac{\pi}{12} + \frac{4\pi}{3}\right)$

$$\cos\frac{\pi}{12},\ \cos\left(\frac{9\pi}{12}\right),\ \cos\left(\frac{17\pi}{12}\right)$$

$\therefore$ Roots of eq. (1) are $4\cos\frac{\pi}{12}$, $4\cos\frac{9\pi}{12}$, $4\cos\frac{17\pi}{12}$

$\frac{\pi}{12} + \frac{16\pi}{12}$

$\frac{17\pi}{12}$

③ $2x^3 + 3x^2 + 3x + 1 = 0$ — ①

$\alpha+\beta+\gamma = -\frac{3}{2}$

Put $y = 2x$.
$\Rightarrow x = \frac{y}{2}$

$= -\frac{1}{2}$

$\frac{2y^3}{2^3} + \frac{3y^2}{2^2} + \frac{3y}{2} + 1 = 0.$

$\frac{y^3 + 3y^2 + 6y + 4}{2^2} = 0$

$y^3 + 3y^2 + 6y + 4 = 0$ — ②

$y = 2x$

$\alpha+\beta+\gamma = -3$

$3\alpha = -3$

$\boxed{\alpha = -1}$

	y^3	y^2	y	c
-1	1	3	6	4
		-1	-2	-4
-1	1	2	4	$\boxed{0}$
		-1	-1	
-1	1	1	$\boxed{3}$	
		-1		
	1	$\boxed{0}$		

$z^3 + 3z = 0.$
$z(z^2+3) = 0$
$z = 0,\ z^2 = -3$
$z = \pm\sqrt{3}i$

$\therefore$ Roots of z are $0, \pm\sqrt{3}i$

$z^3 + 3z = 0$ — ③
Put $z = u+v$
$z^3 - 3uvz - (u^3+v^3) = 0$ — ④
Compare ③ & ④
$-3uv = 3$
$uv = -1$
$u^3v^3 = -1$

$-(u^3+v^3) = 0$
$u^3+v^3 = 0$

$p^2 - 1 = 0$
$p^2 = 1$
$p = \pm 1$

DATE
PAGE

$\therefore u^3 = 1, v^3 = -1$

$u = 1, v = -1$ is the one of the value.

i.e. $z = 0$

Now, find other two roots of z.

$z(z^2+3) = 0$

$z^2 = -3$

$z = \pm\sqrt{3}i$

Roots of eq. ③ is $0, \pm\sqrt{3}i$. $(\alpha+1, \beta+1, \gamma+1)$

Roots of eq. ② is $-1, -1 \pm \sqrt{3}i$. $(\alpha+1-1, \beta+1-1, \gamma+1-1) = \alpha, \beta, \gamma$.

Roots of eq. ① is. $\frac{-1}{2}, \frac{-1 \pm \sqrt{3}i}{2}$

(1) $x^3 - 15x^2 - 33x + 847 = 0$ -①

To Reduce 2nd term we have to divide eq. ① with $x-5$

5	1	-15	-33	847
		5	-50	-415
5	1	-10	-83	$\boxed{432}$
		5	-25	
5	1	-5	$\boxed{-108}$	
		5		
	1	$\boxed{0}$		

$x-5 = y$

$y^3 - 108y + 432 = 0$. -②

$z^3 + 3Hz + G = 0$

$3H = -108$ | $G = 432$

$H = -36$

Put $y = u+v$

$y^3 - 3uvy - (u^3+v^3) = 0$ -③

$G^2 + 4(-36)^3$

$= (432)^2 - 4(36)^3$

$= 0$

DATE ______
PAGE ______

Compare ② & ③

$-3uv = -108$

$uv = 36$

$u^3v^3 = (36)^3$

$u^3 + v^3 = -432$

$\therefore \quad p^2 + 432p + (36)^3 = 0.$

$$p = \frac{-432 \pm \sqrt{(432)^2 - 4(36)^3}}{2}$$

$$\cancel{p = \frac{-432 \pm \sqrt{(2)^2 (36)^2 (3)^6 - 4(36)^3}}{2}}$$

$$\cancel{p = -432 \pm \sqrt{(36)^2 - (36)^3 4}}$$

$$p = \frac{-432 \pm \sqrt{[(4)^2(3)^3]^2 - 4[(4)\times 9]^3}}{2}$$

$$p = \frac{-432 \pm \sqrt{(4)^4(3)^6 - (4)^4(9)^3}}{2}.$$

$$= \frac{-432 \pm \sqrt{(4)^4(3)^6 - (4)^4(3^2)^3}}{2}$$

$$p = \frac{-432}{2}$$

$p = -216$

$u^3 = -216.$

we know $u^3 + v^3 = -432$

$-216 + v^3 = -432.$

$v^3 = -432 + 216$

$v^3 = -216.$

$u = -6$ is one of the value

$v = -6$ is one of the value

$y = u + v \Rightarrow y = -6 - 6 = -12$

DATE
PAGE

Eq. ② $y^3 - 108y + 432 = 0$.

-12 is a root of eq. ②.

$$\begin{array}{r|rrrr} -12 & 1 & 0 & -108 & 432 \\ & & -12 & 144 & -432 \\ \hline & 1 & -12 & 36 & \boxed{0} \end{array}$$

$$\begin{array}{r} 144 \\ -108 \\ \hline 36 \end{array}$$

$y^2 - 12y + 36 = 0$.

$y^2 - 6y - 6y + 36 = 0$.

$y(y-6) - 6(y-6) = 0$

$(y-6)(y-6) = 0$.

$(y-6)^2 = 0$

$\therefore y = 6, 6$.

Roots of Eq. ② is $-12, 6, 6$.

$\therefore$ Roots of Eq. ① is $-7, 11, 11$

Scanned with CamScanner

Methods to solve a Biquadratic Equation:-

(1) Descarte's Method:-

$x^4 - 3x^2 - 42x - 40 = 0.$

$(x^4 - 3x^2 - 42x - 40) = (x^2 + px + q)(x^2 - px + q')$

$= x^4 + (-p^2 + q + q')x^2 + (pq' - pq)x + qq'$

Compare Coeff.

$-p^2 + q + q' = -3 \Rightarrow q + q' = p^2 - 3.$

$pq' - pq = -42. \Rightarrow (q - q') = \frac{42}{p}.$

$qq' = -40$

$(q+q')^2 - (q-q')^2 = 4qq'.$

$(p^2-3)^2 - \left(\frac{42}{p}\right)^2 = 4(-40).$

$p^4 - 6p^2 + 9 - \frac{1764}{p^2} = -160.$

$p^6 - 6p^4 + p^2 - 1764 = -160p^2.$

$p^6 - 6p^4 + 169p^2 - 1764 = 0$

Put $p^2 = t$

$t^3 - 6t^2 + 169t - 1764 = 0$ [illegible]

$t = 9.$

We know, $p^2 = t$

$p^2 = 9$

$p = \pm 3$

We know, $q + q' = p^2 - 3$

$= 9 - 3$

$q + q' = 6.$ — ①

$q - q' = \frac{42}{p}$

$q - q' = \frac{42}{3} = 14.$ — ②

Add ① & ②.

$q + q' = 6$

$q - q' = 14$

$2q = 20$

$\boxed{q = 10}$

Put q in eq. ①

$\boxed{q' = -4}$

$x^4 - 3x^2 - 42x - 40 = (x^2 + px + q)(x^2 - px + q')$

$= (x^2 + 3x + 10)(x^2 - 3x - 4)$

$\left(\frac{-3 \pm \sqrt{9 - 40}}{2}\right)$ $(x^2 - 4x + x - 4)$

$x(x - 4) + (x - 4)$

$(x + 1)(x - 4)$

We get roots i.e. $\frac{-3 \pm \sqrt{31}\,i}{2}$, -1, 4.

Roots of eq ① $-1, \frac{1}{4}$,

$$2x^4+6x^3-3x^2+2=0.$$

Put $y=\frac{1}{x} \Rightarrow x=\frac{1}{y}$

$$\frac{2}{y^4}+\frac{6}{y^3}-\frac{3}{y^2}+2=0.$$

$$2y^4-3y^2+6y+2=0.$$

$$y^4-\frac{3}{2}y^2+3y+1=0.$$

$-p^2+q+q'=-\frac{3}{2}$	$p(q'-q)=3.$
$q+q'=p^2-\frac{3}{2}$	$q-q'=\frac{-3}{p}$

$$qq'=1.$$

$$(q+q')^2-(q-q')^2=4qq'$$

$$\left(p^2-\frac{3}{2}\right)^2-\left(\frac{9}{p^2}\right)=4$$

$$\frac{(2p^2-3)^2}{4}-\frac{9}{p^2}=4.$$

$$\frac{4p^4+9-12p^2}{4}-\frac{9}{p^2}=4.$$

$$p^4 + \frac{9}{4} - \frac{3}{2}p^2 - \frac{9}{p^2} - 1 = 0$$

$$\cancel{8p^6 - 3} \quad \cancel{-72} = 0.$$

$$\cancel{8p^2}$$

$$\frac{4p^6 + 9p^2 - 12p^4 - 36 - 16p^2}{4p^2} = 0$$

$$4p^6 - 7p^2 - 36 - 12p^4 = 0.$$

$$p^2 = 4.$$

$$p = \pm 2.$$

When $p = 2$

$$q + q' = \frac{5}{2}$$

$$q - q' = \frac{3}{2}$$

$$2q = 4$$

$$\boxed{q = 2}$$

$$\boxed{q' = \frac{1}{2}}$$

$$y^4 - \frac{3}{2}y^2 + 3y + 1 = \left(x^2 + 2x + \frac{1}{2}\right)\left(x^2 - 2x + 2\right)$$

$$= \frac{-2 \pm \sqrt{4-2}}{2}, \quad \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{2}}{2}, \quad \frac{2 \pm \sqrt{4}\,i}{2}$$

$= -1 \pm \frac{1}{\sqrt{2}},\ 1 \pm i.$

Roots of eq. (1) is Reciprocal of $-1 \pm \frac{1}{\sqrt{2}},\ 1 \pm i.$

(3) $x^4 - 6x^3 + 3x^2 + 22x - 6 = 0.$ — (1)

Put. $y = 2x. \Rightarrow x = \frac{y}{2}$

$$\left(\frac{y^4}{16}\right) - \frac{6y^3}{8} + \frac{3}{4}y^2 + \frac{22}{2}y - \frac{6}{1} = 0$$

$$\frac{y^4 - 12y^3 + 12y^2 + 176y - 96}{16} = 0.$$

$\Rightarrow y^4 - 12y^3 + 12y^2 + 176y - 96 = 0.$ — (2)

	y^4	y^3	y^2	y	c
3	1	-12	12	176	-96
		3	-27	-45	393
3	1	-9	-15	131	297
		3	-18	-99	
3	1	-6	-33	32	
		3	-9		
3	1	-3	-42		
		3			
	1	0			

$\frac{+123}{4}$

$y - 3 = z$

$z^4 - 42z^2 + 32z + 297 = 0$ — (3)

$(z^2 + pz + q)(z^2 - pz + q') = z^4 - 42z^2 + 32z + 297.$

$-p^2 + q + q' = -42$

$q + q' = p^2 - 42$ — (1)

$$p(q'-q) = 32$$

$$q'-q = \frac{32}{p} \quad \text{—— (7)}$$

$$qq' = 297.$$

$$(q+q')^2 - (q'-q)^2 = 4qq'.$$

$$(p^2-42)^2 - \left(\frac{32}{p}\right)^2 = 4(297)$$

$$p^4 + 1764 - 84p^2 - \frac{1024}{p^2} = 1188.$$

$$p^6 + 1764p^2 - 84p^4 - 1024 = 1188p^2.$$

$$p^6 + 576p^2 - 84p^4 - 1024 = 0.$$

$$p^6 - 84p^4 + 576p^2 - 1024 = 0.$$

Put. $p^2 = t$.

$$t^3 - 84t^2 + 576t - 1024 = 0.$$

$t = 4.$

4 satisfy the eqⁿ. therefore 4 is a root of eq.

$\therefore p^2 = 4.$

DATE
PAGE

$q + q' = 4 - 42$

$q + q' = -38$

$q' - q = \frac{32}{2}$

$q' - q = 16.$

$q + q' = -38$

$q' - q = 16$

$2q' = -22$

$q' = -11$

$q - 11 = -38$

$q = -38 + 11$

$q = -27.$

$z^4 - 42z^2 + 32z + 297 = (z^2 + pz + q)(z^2 - pz + q')$

$= (z^2 + 2z + q)(z^2 - 2z + q')$

$= (z^2 + z + z - 27)(z^2 - 2z - 11)$

$= (z^2 + 2z - 27)(z^2 - 2z - 11)$

$= \frac{-2 \pm \sqrt{4 + 4(27)}}{2}, \frac{2 \pm \sqrt{4 + 4(11)}}{2}$

$= \frac{-2 \pm \sqrt{4(1+27)}}{2}, \frac{2 \pm \sqrt{4(1+11)}}{2}$

$= \frac{-2 \pm 2\sqrt{28}}{2}, \frac{2 \pm 2\sqrt{12}}{2}$

$= -1 \pm \sqrt{28}, \quad 1 \pm 2\sqrt{7}.$

Roots of eq. (3) are $z = -1 \pm 2\sqrt{7}, \quad 1 \pm 2\sqrt{7}.$

Roots of eq. (1) are $y = 2 \pm 2\sqrt{7}, \quad 4 \pm 2\sqrt{7}$

Roots of eq. (1) are $x = 1 \pm \sqrt{7}, \quad 2 \pm \sqrt{3}.$

3^{-1}

3^{+1}

-27
-11
$\overline{38}$

$-11 \quad 27$

38
-16
$\overline{22}$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$11 - q = 16$

$-q = 16 - 11$

38
-11
$\overline{27}$

$-27 - 11$
$= -38$

$-11 + 27$

$3 \times 9 = 27$

$3 \mid 27$
$3 \mid 9$
$3 \mid 3$
$\quad 1$

$x^4 - 2x^3 - 5x^2 + 10x - 3 = 0.$ —(1)

Put $y = 2x \Rightarrow x = \frac{y}{2}$

$\frac{y^4}{2^4} - \frac{y^3}{2^2} - \frac{5y^2}{2^2} + \frac{10y}{2} - 3 = 0.$

i.e. $y^4 - 4y^3 - 20y^2 + 80y - 48 = 0.$ —(2)

1	1	-4	-20	80	-48	
		1	-3	-23	57	
1	1	-3	-23	57	9.	
		1	-2	-25		
1	1	-2	-25	32		
		1	-1			
1	1	-1	-26			
		1				
	1	0.				

$z^4 - 26z^2 + 32z + 9 = 0.$ —(3)

$z^4 - 26z^2 + 32z + 9 = (z^2 + pz + q)(z^2 - pz + q').$

$-p^2 + q + q' = -26$

$q + q' = p^2 - 26$ —(4)

$pq' - pq = 32$

$p(q' - q) = 32$

$q' - q = \frac{32}{p}$ —(5)

$$qq' = 9$$

$$(q+q')^2 - (q'-q)^2 = 4qq'$$

$$(p^2-26)^2 - \left(\frac{32}{p}\right)^2 = 4(9)$$

$$p^4 - 52p^2 + (26)^2 - \frac{(32)^2}{p^2} = 36$$

$$p^6 - 52p^4 + (26)^2 p^2 - (32)^2 = 36p^2$$

$$p^6 - 52p^4 + 676p^2 - 36p^2 - 1024 = 0$$

$$p^6 + 640p^2 - 52p^4 - 1024 = 0$$

Put $p^2 = t$

$$t^3 - 52t^2 + 640t - 1024 = 0$$

Put $t = 16$

$\therefore p^2 = 16, \quad p = \pm 4$

Put p^2 in eq. (4)

$$q + q' = 16 - 26$$

$$q + q' = -10 \quad \text{— (6)}$$

Put p in eq. (5)

$$q' - q = \frac{32}{4}$$

$$q' - q = 8 \quad \text{— (7)}$$

$$q' + q = -10$$

$$q' - q = 8$$

$$2q' = -2$$

$$q' = -1$$

Put q' in eq. (6)

$$q - 1 = -10$$

$$\boxed{q = -9}$$

Put q' in eq. (7).

$$-1 - q = 8$$

$$-1 - 8 = q$$

$$\boxed{-9 = q}$$

$-b \pm \sqrt{b^2-4ac}$
$2a$

$$z^4 - 26z^2 + 32z + 9 = (z^2 + 4z - 9)(z^2 - 4z - 1)$$

$$= \frac{-4 \pm \sqrt{16+36}}{2} ; \frac{4 \pm \sqrt{16+4}}{2}$$

$$= \frac{-4 \pm \sqrt{52}}{2} ; \frac{4 \pm \sqrt{20}}{2}$$

$$= \frac{-4 \pm 2\sqrt{13}}{2} ; \frac{4 \pm 2\sqrt{5}}{2}$$

Roots of eq. ③ is $z = -2 \pm \sqrt{13} ; 2 \pm \sqrt{5}$

Roots of eq. ② is $y = -1 \pm \sqrt{13} ; 3 \pm \sqrt{5}$

Roots of eq. ① is $x = \frac{-1 \pm \sqrt{13}}{2} ; \frac{3 \pm \sqrt{5}}{2}$

——*——*——*——

Ferrari's Methods of Solving a biquadratic.

Example :- $x^4 - 8x^3 + 11x^2 + 20x + 4 = 0.$

$x^4 + 11x^2 - 8x^3 = -20x - 4$ $\qquad + 2\sqrt{11}x^3$

$\cancel{(x^2 + \sqrt{11}x)^2 - 2\sqrt{11}x^3}$ $\qquad 2a_1 = 8,\ a_1 = \frac{8}{2} = 4.$

$(x^2 - 4x)^2 - 16x^2 + 11x^2 = -20x - 4.$

$(x^2 - 4x)^2 \quad - 5x^2 = -20x - 4$

$(x^2 - 4x)^2 = 5x^2 - 20x - 4.$ $\qquad -2\lambda 4x \quad +2\lambda x^2$

$(x^2 - 4x + \lambda)^2 = 5x^2 - 20x - 4 + \lambda^2 + 2\lambda(x^2 - 4x).$

$(x^2 - 4x + \lambda)^2 = (5 + 2\lambda)x^2 - (20 + 8\lambda)x + (\lambda^2 - 4).$ — (*)

R.H.S is a perfect square if, $b^2 - 4ac = 0.$

> Perfect square eq^n has always 0 discriminant if $b^2 - 4ac = 0$

$(-1)^2(20 + 8\lambda)^2 - 4(5 + 2\lambda)(\lambda^2 - 4) = 0$

$(20)^2 + (8\lambda)^2 + 20(8\lambda)2 - (20 + 8\lambda)(\lambda^2 - 4) = 0$

$400 + 64\lambda^2 + 320\lambda - 20\lambda^2 + 80 - 8\lambda^3 - 4\lambda = 0$

$-8\lambda^3 + 44\lambda^2 + 316\lambda + 480 = 0.$

$-4\lambda^3 + 22\lambda^2 + 158\lambda + 240 = 0.$

$-2\lambda^3 + 11\lambda^2 + 79\lambda + 120 = 0.$

Scanned with CamScanner

DATE
PAGE

*
$2\lambda^3 - 11\lambda^2 - 79\lambda - 120 = 0.$

Other Method :-

$(-20-8\lambda)^2 - 4(5+2\lambda)(\lambda^2-4) = 0.$

$[(-4)^2(5+2\lambda)^2 - 4(5+2\lambda)(\lambda^2-4)] = 0.$

$16(5+2\lambda)^2 - 4(\lambda^2-4)(5+2\lambda)] = 0.$

$4(5+2\lambda)\ [4(5+2\lambda) - (\lambda^2-4)] = 0.$

$\lambda = \frac{-5}{2}$

From eq. (*) Put $\lambda = \frac{-5}{2}$

$\left(x^2 - 4x - \frac{5}{2}\right)^2 = -\left(8x\cdot\frac{-5}{2} + 20\right)x + \left(\frac{25}{4} - 4\right)$

$\left(x^2 - 4x - \frac{5}{2}\right)^2 = \frac{9}{4}$

$x^2 - 4x - \frac{5}{2} = \pm\frac{3}{2}$

$x^2 - 4x - \frac{5}{2} - \frac{3}{2} = 0$	$x^2 - 4x - \frac{5}{2} + \frac{3}{2} = 1$
$x^2 - 4x - 4 = 0.$	$x^2 - 4x - 1 = 1.$
$x = \frac{4 \pm \sqrt{16+16}}{2}$	$x = \frac{4 \pm \sqrt{16+4}}{2}$
$x = \frac{4 \pm 4\sqrt{2}}{2}$	$= \frac{4 \pm \sqrt{20}}{2}$

$x = 2 \pm 2\sqrt{2}$; $x = \frac{2 \pm 2\sqrt{5}}{2}$

$= 1 \pm \sqrt{5}$.

$4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$.

$4x^4 - 20x^3 + 33x^2 = 20x - 4$.

$(2x^2 - 5x)^2 - 25x^2 + 33x^2 = 20x - 4$.

$(2x^2 - 5x)^2 + 8x^2 = 20x - 4$.

$(2x^2 - 5x)^2 = -8x^2 + 20x - 4$.

$(2x^2 - 5x)^2 = -(8x^2 - 20x + 4)$.

$(2x^2 - 5x + \lambda)^2 = -(8x^2 - 20x + 4) + \lambda^2 + 2\lambda(2x^2 - 5x)$.

$(2x^2 - 5x + \lambda)^2 = x^2(-8 + 4\lambda) + x(20 - 10\lambda) + \lambda^2 - 4$. — (*)

R.H.S is a perfect square if.

$b^2 - 4ac = 0$.

$(20 - 10\lambda)^2 - 4(-8 + 4\lambda)(\lambda^2 - 4) = 0$

$[10(2 - \lambda)]^2 + 16(2 - \lambda)(\lambda^2 - 4) = 0$.

$(2 - \lambda)[10(2 - \lambda) + 16(\lambda^2 - 4)] = 0$.

$2 - \lambda = 0$

$\boxed{\lambda = 2}$

$20 - 10\lambda$

$+16\lambda^2 - 64$

$16\lambda^2 - 10\lambda - 44$

$10 \pm \sqrt{100 + 64}$

From eq. (*)

$(2x^2-5x+2)^2 = 0,0.$

$2x^2-5x+2 = 0,0.$

$2x^2-4x-x+2=0.$

$2x(x-2)-1(x-2)=0.$

$(2x-1)(x-2)=0$

$x=\frac{1}{2}, x=2$

Roots of eq^n is $\frac{1}{2}, \frac{1}{2}, 2, 2.$

Example ①. $x^4+12x-5=0.$

$(x^2)^2 = -12x+5.$

$(x^2+\lambda)^2 = -12x+5+\lambda^2+2\lambda(x^2).$

$(x^2+\lambda)^2 = 2\lambda x^2 - 12x + \lambda^2+5$ — (*)

RHS is perfect sq. if $b^2-4ac=0.$

$144 - 8\lambda(\lambda^2+5)=0.$

$144-8\lambda^3+40\lambda=0$

$-8\lambda^3+40\lambda+144=0$

$8\lambda^3-40\lambda-144=0.$

$8(\lambda^3 - 5\lambda - 16) = 0$

$\lambda^3 - 5\lambda - 18 = 0$.

$\lambda = 2$

From eq. (*) Put $\lambda = 2$.

$(x^2+2)^2 = 4x^2 - 12x + 9$.

$(x^2+2)^2 = (2x-3)^2$

$1 \pm 2i$
$\sqrt{2}$
$-1 \pm$

$x^2 + 2 = 2x - 3$. ; $x^2 + 2 = -2x + 3$.

$x^2 - 2x + 5 = 0$

$= \dfrac{2 \pm \sqrt{4 - 20}}{2}$

$= \dfrac{2 \pm \sqrt{-16}}{2}$

$= \dfrac{2 \pm 4i}{2}$

$= 1 \pm 2i$.

$x^2 + 2x - 1 = 0$.

$\dfrac{-2 \pm \sqrt{4+4}}{2}$

$= \dfrac{-2 \pm 2\sqrt{2}}{2}$

$= -1 \pm \sqrt{2}$.

$2(2x^4+6x^3-3x^2+2)=0.$

$4x^4+12x^3-6x^2+4=0.$

$(2x^2+3x)^2-9x^2-6x^2=-4.$

$(2x^2+3x)^2-15x^2=-4$

$(2x^2+3x)^2=15x^2-4.$

$(2x^2+3x+\lambda)^2=15x^2-4+\lambda^2+2\lambda(2x^2+3x).$

$(2x^2+3x+\lambda)^2=(15+4\lambda)x^2+6\lambda x+\lambda^2-4.$ — (*)

R.H.S is perfect square if $b^2-4ac=0$.

$(6\lambda)^2-4(15+4\lambda)(\lambda^2-4)=0.$

$36\lambda^2-(60+16\lambda)(\lambda^2-4)=0.$ $-[60\lambda^2-240+16\lambda^3-64\lambda]$

$36\lambda^2-60\lambda^2+240-16\lambda^3+64\lambda=0.$

$-16\lambda^3-24\lambda^2+64\lambda+240=0.$

$16\lambda^3+24\lambda^2-64\lambda-240=0$

$4(4\lambda^3+6\lambda^2-16\lambda-60)=0$

~~$\lambda^3-3\lambda^2+4\lambda+15=0$~~ $2(2\lambda^3+3\lambda^2-8\lambda-30)=0.$

↓ 1, 3, 5, 15

$2\lambda^3+3\lambda^2-8\lambda-30=0$

1, 2, 3, 5, 6, 10, 15, 30

~~$\lambda^3+3\lambda^2-4\lambda-15=0$~~

(3) $x^3 + 12x + 12 = 0$ — (1)

By Cardon's Method

Put $x = u+v$

$x^3 - 3uvx - (u^3+v^3) = 0$ — (2)

$12 = 3 \cdot 4$

Compare (2) & (1).

$-3uv = 12$ $\qquad$ $-(u^3+v^3) = 12$

$uv = -4$ $\qquad$ $u^3+v^3 = -12$

$u^3v^3 = -64$

$p^2 + 12p - 64 = 0$

$p = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2}$

$= \dfrac{-12 \pm \sqrt{(12)^2 - 4(1)(-64)}}{2}$

$= \dfrac{-12 \pm \sqrt{144 + 4(64)}}{2}$

$= \dfrac{-12 \pm \sqrt{4(36) + 4(64)}}{2}$ ✓

$= \dfrac{-12 \pm \sqrt{4(36+64)}}{2}$

$= \dfrac{-12 \pm \sqrt{4(100)}}{2}$

$= \dfrac{-12 \pm \sqrt{400}}{2}$

$= \dfrac{-12 \pm 20}{2}$

$p = -6 \pm 10$

$p = -6 - 10 = -16$ $\qquad$ $p = -6 + 10 = 4$

Scanned with CamScanner

$p = 4, -16$

$\therefore u^3 = 4 \Rightarrow u = (4)^{1/3}$

$v^3 = -16, \Rightarrow v = (-16)^{1/3}$.

$4 = r\cos\theta + ir\sin\theta$.

$r\cos\theta = 4$ — ①

$r\sin\theta = 0$. — ②

squaring & Adding eq. ① & ②

$r^2\cos^2\theta + r^2\sin^2\theta = (4)^2 + (0)^2$

$r^2(\cos^2\theta + \sin^2\theta) = (4)^2$

$r^2 = (4)^2$

$r = 4$.

$r = \sqrt{(0)^2 + (4)^2}$

$= \sqrt{(4)^2}$

$r = 4$

$\theta = \tan^{-1}\frac{y}{x}$

$= \tan^{-1}\frac{0}{4}$

$\theta = \tan^{-1}(0)$

$\tan\theta = 0$.

$\tan\theta = \tan 0°$

$\theta = 0°$.

Put r in eq. ① & ②

$4\cos\theta = 4$

$\cos\theta = 1$

$\cos\theta = \cos 0°$

$\theta = 0°$

$4\sin\theta = 0$

$\sin\theta = 0$.

$\sin\theta = \sin n\pi$

when $n = 0$

$\theta = n\pi$

$\theta = 0°$

[θ lie in 1st quadrant]

$\therefore 4\cos 0° + i4\sin 0°$.

$4\,Cis\,0°$

$(4)^{1/3} = (4\,Cis\,0)^{1/3}$

$= (4)^{1/3}(Cis\,0°)^{1/3}$.

$= (4)^{1/3}(Cis\,2n\pi + 0°)^{1/3}$ $\quad n = 0, 1, 2.$

$= (4)^{1/3}\left(Cis\,\frac{2n\pi + 0°}{3}\right)$

$= (4)^{1/3}\left(Cis\,\frac{2n\pi}{3}\right)$

$= 4^{1/3}(Cis\,0°)$ $\quad$ [we know $Cis\,0° = 1$]

$= 4^{1/3}(1)$

$= (4)^{1/3}$.

Scanned with CamScanner

PAGE

$$(4)^{1/3}\left(Cis\,\frac{2n\pi}{3}\right) = (4)^{1/3}\left(Cis\,\frac{2n\pi + 0}{3}\right)$$ n=1

$$= (4)^{1/3}\left(Cis\,\frac{2\pi}{3}\right)$$ (n=1

$$= (4)^{1/3}\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$= (4)^{1/3}\left[\cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= (4)^{1/3}\left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$

$$= (4)^{1/3}\left[\frac{-1 + \sqrt{3}i}{2}\right].$$

When n = 2

$$(4^{1/3})\left(Cis\,\frac{2n\pi}{3}\right) = (4)^{1/3}\left(Cis\,\frac{4\pi}{3}\right)$$

$$= (4)^{1/3}\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right]$$

$$= 4^{1/3}\left[\cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)\right]$$

$$= 4^{1/3}\left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

$$= (4)^{1/3}\left[\frac{-1 - \sqrt{3}i}{2}\right]$$

Scanned with CamScanner

DATE
PAGE

$(4)^{1/3}\left(cis \frac{2n\pi}{3}\right) = (4)^{1/3}(1), (4)^{1/3}\left(\frac{-1+\sqrt{3}i}{2}\right), (4)^{1/3}\left(\frac{-1-\sqrt{3}i}{2}\right)$

$= 4^{1/3}, (4)^{1/3}w^2, (4)^{1/3}w.$

$(-16)^{1/3} = v.$

$-16 = r\cos\theta + ir\sin\theta.$

$r\cos\theta = -16$ $\quad$ $r^2\cos^2\theta = 256$ — ①

$r\sin\theta = 0$ $\quad$ $r^2\sin^2\theta = 0.$ — ②

Add eq ① & ②

$r^2(\cos^2\theta + \sin^2\theta) = 256$

$r = 16$

$r\cos\theta = -16$	$r\sin\theta = 0$
$16\cos\theta = -16$	$\sin\theta = 0.$
$\cos\theta = -1$	

θ lie in 2nd quadrant

$\cos\theta = -\cos 0°$

$\cos\theta = \cos(180 - 0°)$

$\cos\theta = \cos 180°$

$\theta = 180°$

$\boxed{\theta = \pi}$

$(16\, cis\, \pi)^{1/3} \Rightarrow (16)^{1/3}\left[cis \frac{2n\pi + \pi}{3}\right] \quad n = 0, 1, 2.$

$= (16)^{1/3}\left[cis \frac{\pi}{3}\right], (16)^{1/3}(cis\, \pi), (16)^{1/3}\left(cis \frac{5\pi}{3}\right)$

$= (16)^{1/3}(-1), (16)^{1/3}\left[\overset{cis\, \pi/3}{\frac{1+i\sqrt{3}}{2}}\right], (16)^{1/3}\left[\overset{cis\, 5\pi/3}{\frac{1-\sqrt{3}i}{2}}\right]$

$= (-16)^{1/3}, (16)^{1/3}\left[\frac{(-1)(-1-\sqrt{3}i)}{2}\right], (16)^{1/3}\left[(-1)\left(\frac{-1+\sqrt{3}i}{2}\right)\right]$

$= (-16)^{1/3}, (16)^{1/3} - \omega, (16)^{1/3} - \omega^2$

$= (-16)^{1/3}, (-16)^{1/3}\omega, (16)^{1/3}\omega^2.$

$a^{1/3}, a^{2/3}\omega$

DATE
PAGE

We got $u = (4)^{1/3}, (4)^{1/3}\omega, (4)^{1/3}\omega^2$.

$v = (-16)^{1/3}, (-16)^{1/3}\omega, (-16)^{1/3}\omega^2$.

$\therefore x = (4)^{1/3} - (16)^{1/3}$.

$x = \omega[(4)^{1/3} - (16)^{1/3}]$

$x = \omega^2[(4)^{1/3} - (16)^{1/3}]$

(2) $x^3 - 6x - 9 = 0 \quad -①$

By Cardon's method.

Put $x = u+v$

$x^3 - 3uvx - (u^3+v^3) = 0 - ②$.

Compare ① & ②

$-3uv = -6$

$uv = 2$

$u^3v^3 = 8$

$-(u^3+v^3) = -9$

$u^3+v^3 = 9$

$-6a = 3H$

$H = -2$

$G = -9$

$81 + 4(-2)^3$

$81 - 32$

49

$p^2 - 9p + 8 = 0$.

$a = 1$

$b = -9$

$c = 8$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$$= \frac{9 \pm \sqrt{81 - 4(1)(8)}}{2}$$

$$= \frac{9 \pm \sqrt{81 - 32}}{2}$$

$$p = \frac{9 \pm 7}{2}$$

$$p = \frac{9-7}{2} = 1. \qquad p = \frac{9+7}{2} = 8.$$

Scanned with CamScanner

$u^3 = 1 \Rightarrow u = 1$ is one of the value of $(1)^{1/3}$
$v^3 = 8 \Rightarrow v = 2$ is one of the value

$x = u+v$
$= 1+2$
$= 3.$

	x^3	x^2	x	c
3	1	0	-6	-9
		3	9	9
	1	3	$+3$	$\boxed{0}$

$x^2 + 3x + 3 = 0.$

$$x = \frac{-3 \pm \sqrt{9-12}}{2}$$

$$= \frac{-3 \pm \sqrt{-3}}{2}$$

$$= \frac{-3 \pm \sqrt{3}i}{2}$$

$\frac{-3-\sqrt{3}i}{2}, \frac{-3+\sqrt{3}i}{2}$

Roots of eq. ① is $3, \frac{-3 \pm \sqrt{3}i}{2}$

$\frac{(-3 \pm 3i)^3}{(2)^3}$

Use Descarte's method to solve

④ $x^4 - 15x^2 + 20x - 6 = 0.$ — ①

$(x^2+px+q)(x^2-px+q')$

$(x^2+px+q)(x^2-px+q') = 0$ — ②

Compare eq. ① & ②.

$-p^2 + (q+q') = -15.$

$q+q' = p^2 - 15.$ — ③

$p(q'-q) = 20$

$q'-q = \frac{20}{p}.$ — ④

$qq' = -6.$

$(q+q')^2 - (q'-q)^2 = 4qq'$

$\left(p^2-15\right)^2 - \left(\frac{20}{p}\right)^2 = 4(-6)$

$= -24$

$p^4 + (15)^2 - 30p^2 - \frac{400}{p^2}$

$p^6 + 225p^2 - 30p^4 - 400 = -24p^2$

$p^6 - 30p^4 + 249p^2 - 400 = 0$

put $p^2 = t$

$t^3 - 30t^2 + 249t - 400 = 0$

$t = 16$

$\therefore p^2 = 16$

$p = \pm 4$

Eq. (3) $q + q' = p^2 - 15$

$= 16 - 15$

$q + q' = 1 \quad -(5)$

Eq. (4) $q' - q = \frac{20}{p}$

when $p = 4$

$q' - q = \frac{20}{4}$

$q' - q = 5 \quad -(6)$

Add (5) & (6)

$2q' = 6$

$q' = 3$

DATE
PAGE

Put q' in eq. (6).

$3 - q = 5$

$3 - 5 = q$

$-2 = q$

-2

$x^4 - 15x + 20x - 6 = (x^2 + px + q)(x^2 - px + q')$

$= (x^2 + 4x - 2)(x^2 - 4x + 3)$

$= \frac{-4 \pm \sqrt{16+8}}{2}$, $[x^2 - 3x - x + 3]$

$= \frac{-4 \pm \sqrt{24}}{2}$, $[x(x-3) - 1(x-3)]$

$= \frac{-4 \pm \sqrt{2 \times 2 \times 3 \times 2}}{2}$, $(x-1)(x-3)$

$= \frac{-4 \pm 2\sqrt{6}}{2}$, $\therefore x = 1, 3$

$= -2 \pm \sqrt{6}$,

x^5

$(x^2 + \quad - x)$

Scanned with CamScanner

Use Ferrari's Method to solve

(4) $x^4 - 5x^3 + 3x^2 + 2x + 8 = 0.$

$$\left(x^2 - \frac{5}{2}x\right)^2 - \frac{25x^2}{4} + 3x^2 + 2x + 8 = 0.$$

$$\left(x^2 - \frac{5}{2}x\right)^2 - \frac{13x^2}{4} + 2x + 8 = 0.$$

$$\left(x^2 - \frac{5}{2}x\right)^2 = \frac{13x^2}{4} - 2x - 8.$$

$$\left(x^2 - \frac{5}{2}x + \lambda\right)^2 = \frac{13x^2}{4} - 2x - 8 + \lambda^2 + 2\lambda\left(x^2 - \frac{5}{2}x\right)$$

$$\left(x^2 - \frac{5x}{2} + \lambda\right)^2 = \left(\frac{13}{4} + 2\lambda\right)x^2 - (2 + 5\lambda)x + \lambda^2 - 8 \quad — (*)$$

R.H.S is perfect square if $b^2 - 4ac = 0.$

$$(-1)^2(2+5\lambda)^2 - 4\left(\frac{13}{4} + 2\lambda\right)(\lambda^2 - 8) = 0$$

$$4 + 25\lambda^2 + 20\lambda - 4\left[\frac{13 + 8\lambda}{4}\right)(\lambda^2 - 8) = 0.$$

$$4 + 25\lambda^2 + 20\lambda - (13 + 8\lambda)(\lambda^2 - 8) = 0.$$

$$4 + 25\lambda^2 + 20\lambda - [13\lambda^2 - 104 + 8\lambda^3 - 64\lambda] = 0.$$

$$4 + 25\lambda^2 + 20\lambda - 13\lambda^2 + 104 - 8\lambda^3 + 64\lambda = 0.$$

$$-8\lambda^3 + 12\lambda^2 + 108 + 84\lambda = 0.$$

$$-8\lambda^3 + 12\lambda^2 + 84\lambda + 108 = 0.$$

$$-8\left[\lambda^3 - \frac{12\lambda^2}{8} - \frac{84\lambda}{8} - \frac{108}{8}\right] = 0.$$

$$\lambda^3 - \frac{3}{2}\lambda^2 - \frac{21}{2}\lambda - \frac{27}{2} = 0.$$

Put $y = 2\lambda$

$\Rightarrow \lambda = \frac{y}{2}$

$$\frac{y^3}{2^3} - \frac{3y^2}{2^3} - \frac{21y}{2^2} - \frac{27}{2} = 0$$

$$\frac{y^3 - 3y^2 - 42y - (2^2)27}{2^3} = 0$$

$$y^3 - 3y^2 - 42y - 108 = 0.$$

$\therefore y = 9.$

$\Rightarrow \boxed{\lambda = \frac{9}{2}}$

From eq. (*)

$$\left(x^2 - \frac{5}{2}x + \frac{9}{2}\right)^2 = \left[\frac{13}{4} + 2\left(\frac{9}{2}\right)\right]x^2 - \left[2 + 5\left(\frac{9}{2}\right)\right]x + \frac{81}{4} - 8.$$

$$= \left[\frac{13}{4} + 9\right]x^2 - \left[\frac{4+45}{2}\right]x + \frac{81-32}{4}$$

$$= \left[\frac{13+36}{4}\right]x^2 - \frac{49}{2}x + \frac{49}{4}$$

$$= \frac{49}{4}x^2 - \frac{49}{2}x + \frac{49}{4}$$

$$= \frac{49}{2}\left[\frac{x^2}{2} - x + \frac{1}{2}\right]$$

$$= \frac{49}{2}\left[\frac{x^2-2x+1}{2}\right]$$

$$= \frac{49}{4}(x^2-2x+1)$$

$$\left(x^2 - \frac{5x}{2} + \frac{9}{2}\right)^2 = \frac{49}{4}(x-1)^2$$

$$\left(\frac{2x^2-5x+9}{2}\right)^2 = \frac{49}{4}(x-1)^2$$

$$(2x^2-5x+9)^2\,\frac{1}{4} = \frac{49}{4}(x-1)^2$$

$$(2x^2-5x+9)^2 = 49(x-1)^2$$

$$2x^2-5x+9 = \pm\sqrt{49(x-1)^2}$$

$$2x^2-5x+9 = \pm 7(x-1).$$

1st Case	2nd Case
When $2x^2-5x+9 = 7(x-1)$.	When $2x^2-5x+9 = -7(x-1)$
$2x^2-5x+9 = 7x-7$.	$2x^2-5x+9 = -7x+7$.
$2x^2-5x-7x+9+7=0$.	$2x^2-5x+7x+9-7=0$.
$2x^2-12x+16=0$	$2x^2+2x+2=0$.
$2(x^2-6x+8)=0$.	$2(x^2+x+1)=0$
$x^2-6x+8=0$	$x^2+x+1=0$.
$x^2-4x-2x+8=0$	$a=1, b=1, c=1$
$x(x-4)-2(x-4)=0$	$x = \frac{-1\pm\sqrt{1-4(1)(1)}}{2}$
$(x-2)(x-4)=0$	
$x=4,2$	$= \frac{-1\pm\sqrt{3}i}{2}$

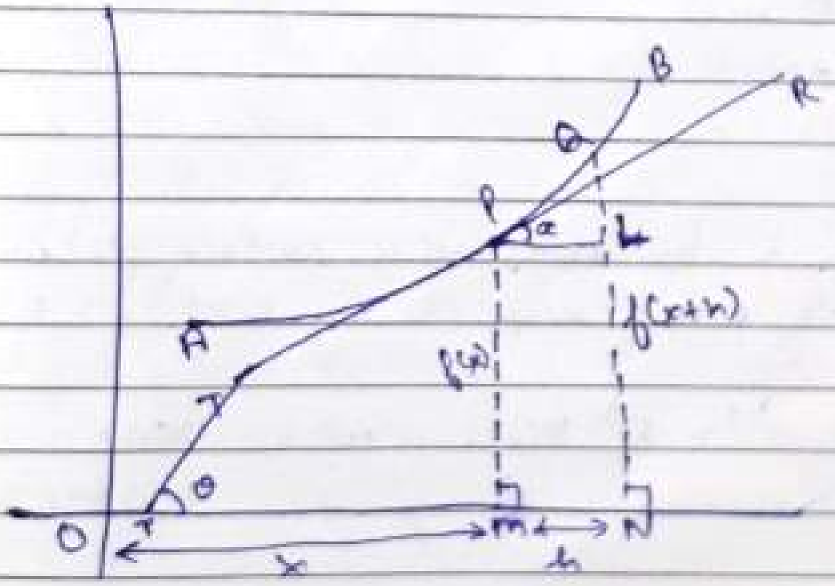
Root of given eqn is $1, 2, \frac{-1 \pm \sqrt{7}i}{2}$.

Properties of derived function

Taylor's Theorem:- Let $f(x)$ be a function defined on an interval $[a, a+h]$ s.t. $f, f', f'', \ldots, f^{(n-1)}$ all exist and $f^{(n)}$ exist in $[a, a+h]$ then.

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots$$

Property 1:- Let $f(x)$ be a polynomial function then $f'(x)$ at any point P = slope of tangent at point P.



Consider a curve APB of fn $y = f(x)$ let PT be the tangent to $f(x)$ at pt. P.

Let $OM = x$, $MN = h$.

then $ON = OM + MN = x + h$; $PM = f(x)$; $QN = f(x+h)$

$f(x+h) - f(x) = QN - PM$

$= QN - LN$

$= QL$

$$\Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{QL}{PL} = \tan\alpha = \tan\angle QPL$$

as $Q \to P$, $h \to 0$ and $\alpha \to \theta$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \tan\theta$$

$$\Rightarrow f'(x) = \tan\theta.$$

Rolle's Theorem

→ 2nd Property :- Between any two consecutive roots of $f(x)$, there exist atleast one root of $f'(x)$.

Ex:- $f(x) = (x-4)(x-6)$

$f'(x) = x-4+x-6.$

$= 2x-10$

Proof:- Let a & b be consecutive roots of $f(x)$ with multiplicity p & q resp. Clearly $p \geq 1$, $q \geq 1$.

$f(x) = (x-a)^p (x-b)^q \psi(x)$. where $\psi(a) \neq 0$

$\psi(b) \neq 0$.

also $\psi(a)$ & $\psi(b)$ have same sign.

$\because$ if $\psi(a)$ & $\psi(b)$ have opp. sign then by intermediate value Theorem, $\exists\, c \in (a,b)$ s.t.

$\psi(c) = 0 \Rightarrow f(c) = 0.$

This is a Contradiction as a & b are consecutive roots of $f(x)$.

hence $\Psi(a)$ & $\Psi(b)$ have same sign

now $f'(x) = p(x-a)^{p-1}(x-b)^q \Psi(x) + (x-a)^p q(x-b)^{q-1}\Psi(x)$

$+ (x-a)^p (x-b)^q \Psi'(x)$.

$= (x-a)^{p-1}(x-b)^{q-1}[p\Psi(x)(x-b) + q(x-a)\Psi(x) + (x-a)(x-b)\Psi'x]$. --- (1)

$S(x) = p(x-b)\Psi x + q(x-a)\Psi(x) + (x-a)(x-b)\Psi'(x)$.

$S(a) = p(a-b)\Psi(a)$.

$S(b) = q(b-a)\Psi(b)$.

$= -q\cdot(a-b)\Psi(b)$.

As p, q are +ve, $\Psi(a)$ & $\Psi(b)$ are of same sign this means $S(a)$ & $S(b)$ are of opposite sign.

$\Rightarrow$ By Intermediate Value theorem $\exists$ some $d \in (a,b)$ s.t. $S(d) = 0 \Rightarrow f'(d) = 0$ from (1)

3rd Property: Let $a_1, a_2, \ldots a_n$ be all n roots of Polynomial $f(x)$, then

$$f'(x) = \frac{f(x)}{x-a_1} + \frac{f(x)}{x-a_2} + \cdots + \frac{f(x)}{x-a_n}$$

Proof: $f(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$.

Replace x by $y+x$.

$f(y+x) = (y+x-a_1)(y+x-a_2)\ldots(y+x-a_n)$

Scanned with CamScanner

$$f(y+x) = (y+\underbrace{x+a_1}_{b_1})(y+\underbrace{x+a_2}_{b_2}) \dots (y+\underbrace{x+a_n}_{b_n})$$

$$= (y+b_1)(y+b_2) \dots (y+b_n).$$

$$= 1y^n + q_1 y^{n-1} + q_2 y^{n-2} + \dots + q_{n-1}y + q_n$$

Sum of roots $= -q_1 = -(b_1+b_2+b_3 \dots + b_n).$

$$= -[x-a_1 + x-a_2 + \dots + x-a_n].$$

$$q_1 = (x-a_1)+(x-a_2)+(x+a_3)+ \dots + (x-a_n).$$

Similarly.

$q_2 =$ ~~$(x-a_1)(x-a_2)+(x-a_2)(x-a_3)$~~

$$\Rightarrow q_2 = \sum_{i\neq j} (x-a_i)(x-a_j).$$

$$-q_3 = \sum_{i\neq j\neq k} [-(x-a_i)][-(x-a_j)][-(x-a_k)].$$

$$\Rightarrow q_3 = \sum_{i\neq j\neq k} [(x-a_i)(x-a_j)(x-a_k)].$$

& so on.

$$q_n = (x-a_1)(x-a_2)(x-a_3) \dots (x-a_n).$$

DATE
PAGE

By Taylor's expansion.

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2} f''(x) + \dots$$

Put $h = y$

(1) $$f(x+y) = f(x) + y f'(x) + \frac{y^2}{2!} f''(x) + \dots \quad -(2)$$

From (1) & (2)

$f'(x) = q_{n-1}$ [q_{n-1} is the coeff. of y in eq. (1)]

$$f'(x) = (x-a_1)(x-a_2)\dots(x-a_{n-1}) + (x-a_1)(x-a_2)\dots(x-a_{n-2})(x-a_n) + (x-a_2)(x-a_3)\dots(x-a_n).$$

$$= \frac{f(x)}{x-a_n} + \frac{f(x)}{x-a_{n-1}} + \frac{f(x)}{x-a_{n-2}} + \dots + \frac{f(x)}{x-a_1}$$

4th Property :- If $x = \alpha$ is a root of $f(x)$ of order m, then $x = \alpha$ is a root of $f'(x)$ of order $m-1$.

5th Property :- If α is a multiple root of $f(x)$ iff α is common root of $f(x)$ & $f'(x)$.

6th Property :- If α is multiple root of $f(x)$, then α is a root of $d(x)$, where $d(x) = \gcd[f(x), f'(x)]$ & vice-versa

DATE
PAGE

~~Theorem~~

Property A:- If $x = \alpha_1$ is a root of $f(x)$ of order m, then $x = \alpha_1$ is a root of $f'(x)$ of order $m-1$.

:-

If $f(x)$ has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ then

$$f'(x) = \frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_2} + \frac{f(x)}{x-\alpha_3} + \cdots + \frac{f(x)}{x-\alpha_n}$$

$$= \underbrace{\frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_1} + \cdots + \frac{f(x)}{x-\alpha_1}}_{m \text{ times}} + \frac{f(x)}{x-\alpha_{m+1}} + \cdots + \frac{f(x)}{x-\alpha_n}$$

$$f'(x) = \frac{m f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_{m+1}} + \cdots + \frac{f(x)}{x-\alpha_n} \quad -①$$

As α_1 is a root of $f(x)$ of order m.

$f(x) = (x-\alpha_1)^m \phi(x)$ where $\phi(\alpha_1) \neq 0$. $\longrightarrow ②$

Put $f(x)$ in eq. ①

$$f'(x) = \frac{m(x-\alpha_1)^m \phi(x)}{(x-\alpha_1)} + \frac{(x-\alpha_1)^m \phi(x)}{x-\alpha_{m+1}} + \cdots + \frac{(x-\alpha_1)^m \phi(x)}{x-\alpha_n}$$

$$= m(x-\alpha_1)^{m-1}\phi(x) + \frac{(x-\alpha_1)^m \phi(x)}{x-\alpha_{m+1}} + \cdots + \frac{(x-\alpha_1)^m \phi(x)}{x-\alpha_n}$$

$$= (x-\alpha_1)^{m-1}\left[m\phi(x) + \frac{(x-\alpha_1)\phi(x)}{x-\alpha_{m+1}} + \cdots + \frac{(x-\alpha_1)\phi(x)}{x-\alpha_n}\right]$$

From eq. (2)

As $\alpha_{m+1}, \alpha_{m+2}, \alpha_{m+3}, \ldots, \alpha_n$ are roots of $f(x)$, so $\alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_n$ are root of $\phi(x)$.

$$\Rightarrow \phi(x) = (x - \alpha_{m+1})\,\phi_1(x).$$
$$\phi(x) = (x - \alpha_{m+2})\,\phi_2(x).$$
$$\vdots$$
$$\phi(x) = (x - \alpha_n)\,\phi_{n-m}(x).$$

of $f(x)$
$x - \alpha_n$.

From eq. (1)

$$f'(x) = (x-\alpha_1)^{m-1}\left[m\phi(x) + (x-\alpha_1)\phi(x) + (x-\alpha_1)\phi_2(x) + \ldots + (x-\alpha_1)\phi_{n-m}(x)\right.$$
$$f'(x) = (x-\alpha_1)^{m-1}\, g(x). \quad — (4)$$
$$\Rightarrow (x-\alpha_1)^{m-1} \mid f'(x).$$

$\Rightarrow$ α_1 is a root of $f'(x)$. of order atleast $m-1$.

If α_1 is a root of $f'(x)$ of order $> m-1$. [from eq. (4)].

$\Rightarrow$ α_1 is a root of $g(x)$

$\Rightarrow g(\alpha_1) = 0$

but $g(\alpha_1) = m\phi(\alpha_1) \neq 0$ $\quad [\because \phi(\alpha_1) \neq 0]$

hence α_1 is a root of $f'(x)$ of order $m-1$.

Unit:- 3

Imp

Newton's Theorem on Sum of Integral powers of roots of a Polynomial:-

Statement:- Consider the Polynomial equation $f(x)$:

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$$

and α_i's be roots of $f(x)$.

Let $S_1 = \Sigma\alpha$, $S_2 = \Sigma\alpha^2$, $S_3 = \Sigma\alpha^3, \dots, S_m = \Sigma\alpha^m$.

$$a_0 S_1 + a_1 = 0$$
$$a_0 S_2 + a_1 S_1 + 2a_2 = 0.$$
$$a_0 S_3 + a_1 S_2 + a_2 S_1 + 3a_3 = 0.$$
$$\vdots$$
$$a_0 S_p + a_1 S_{p-1} + \dots + a_{p-1} S_1 + p a_p = 0.$$

where $a_p = 0$ if $p > n$.

Examples:-

(1) Consider the Polynomial $P(x) = x^3 + 3x^2 + 4x - 8$.
Let α, β, γ be roots of $P(x)$.
Find (1) $\Sigma\alpha^2$.
(2) $\Sigma\alpha^4$.

Compare with $a_0 x^3 + a_1 x^2 + a_2 x + a_3$

Here
$a_0 = 1$
$a_1 = 3$
$a_2 = 4$
$a_3 = -8$.

$$\Sigma\alpha^2 = a_0 S_2 + a_1 S_1 + 2a_2 = 0$$
$$= (1)S_2 + (3)S_1 + 2(4) = 0$$
$$= S_2 + 3S_1 + 8 = 0$$
$$S_2 = -3S_1 - 8$$
$$= 9 - 8$$
$$= 1$$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

(2) $\Sigma\alpha^3 =$

$a_0S_3 + a_1S_2 + a_2S_1 + 3a_3 = 0.$

$S_3 + 3S_2 + 4S_1 + 3(-8) = 0$

$S_3 = -3S_2 - 4S_1 + 24$

$= -3 + 12 + 24$

$= 36 - 3$

$= 33$

(3) $\Sigma\alpha^4 =$

$a_0S_4 + a_1S_3 + a_2S_2 + a_3S_1 = 0.$

$S_4 = -99 - 4 - 24.$

$= -103 - 24$

$= -127.$

$$\begin{array}{r} 76 \\ 20 \\ \hline 96 \end{array}$$

-99
-4

(3) $\Sigma\alpha^{-2} = \Sigma\frac{1}{\alpha^2}$

Put $x = \frac{1}{y}$

$\frac{1}{y^3} + \frac{3}{y^2} + \frac{4}{y} - 8 = 0.$

$1 + 3y + 4y^2 - 8y^3 = 0.$

$-8y^3 + 4y^2 + 3y + 1 = 0.$ — (2).

Let a, b, c be roots of (2).

then $\Sigma a^2 = \Sigma\alpha^{-2}$

Scanned with CamScanner

$a_0 = -8$

$a_1 = 4$

$a_2 = 3$

$a_3 = 1.$

$S_1 = \frac{-4}{-8} = \frac{1}{2}$

$-8S_2 = -\frac{4}{2} - 2(3)$

$-8S_2 = -2 - 6.$

$-8S_2 = -8$

$\boxed{S_2 = 1.}$

$\therefore \Sigma a^2 = 1.$

$\boxed{\Sigma \alpha^{-2} = 1.}$

$a_0 S_3 = -a_1 S_2 - a_2 S_1 - 3a_3$

$-8S_3 = -4(1) - 3\left(\frac{1}{2}\right) - 3(1)$

$-8S_3 = -4 - \frac{3}{2} - 3.$

$-8S_3 = \frac{-8-3-6}{2}$

$-16S_3 = -17$

$S_3 = \frac{17}{16}$

(2) $\Sigma \alpha^{-4} = \Sigma a^4.$

$a_0 S_4 = -a_1 S_3 - a_2 S_2 - a_3 S_1.$

$-8S_4 = -4$

DATE
PAGE

Example: Evaluate $(1+\sqrt{5}i)^{10} + (1-\sqrt{5}i)^{10}$.

Let $\alpha = 1+\sqrt{5}i$; $\beta = 1-\sqrt{5}i$.

$\alpha+\beta = 1+\sqrt{5}i + 1 - \sqrt{5}i$
$= 2$

$\alpha\beta = (1+\sqrt{5}i)(1-\sqrt{5}i)$
$= (1)^2 - (\sqrt{5}i)^2$
$= 1+5$
$= 6.$

We construct a quadratic eq^n whose roots are α & β.

$x^2 - 2x + 6 = 0.$ $\qquad a_0x^2 + a_1x + a_2 = 0.$

then $(1+\sqrt{5}i)^{10} + (1-\sqrt{5}i)^{10} = S_{10}$ of above eq^n.

$S_1 = 2$

$a_0S_2 = -a_1S_1 - 2a_2$
$= 2(2) - 2(6).$
$= 4 - 12$

$\boxed{S_2 = -8}$

$a_0S_3 = -a_1S_2 - a_2S_1 - 3a_3.$
$= -(-2)(-8) - 6(2).$
$= -16 - 12.$

$\boxed{S_3 = -28}$

$a_0S_4 = -a_1S_3 - a_2S_2.$
$= -(-2)(-28) - (6)(-8)$
$= -56 + 48.$

$\boxed{S_4 = -8}$

16
12
28

168
-56
112

28
6
168

28
2
56

$$a_0 S_5 = -a_1 S_4 - a_2 S_3$$
$$= -(-2)(-8) - (6)(-28)$$
$$= -16 + 168$$
$$\boxed{S_5 = 152}$$

$$a_0 S_6 = -a_1 S_5 - a_2 S_4$$
$$= -(-2)(152) - (6)(-8)$$
$$= 304 + 48$$
$$\boxed{S_6 = 352}$$

$$a_0 S_7 = -a_1 S_6 - a_2 S_5$$
$$= -(-2)(352) - (6)(152)$$
$$= 704 - 912$$
$$\boxed{S_7 = -208.}$$

$$a_0 S_8 = -a_1 S_7 - a_2 S_6$$
$$= -(-2)(-208) - (6)(352)$$
$$= -416 - 2112$$
$$\boxed{S_8 = -2528}$$

$$a_0 S_9 = -a_1 S_8 - a_2 S_7$$
$$= -(-2)(-2528) - (6)(-208)$$
$$= -5056 + 1248$$
$$\boxed{S_9 = -3808.}$$

$$a_0 S_{10} = -a_1 S_9 - a_2 S_8$$
$$= -(-2)(-3808) - (6)(-2528)$$
$$= -7616 + 15168$$
$$\boxed{S_{10} = 7552}$$

3808
3808
7616

2528
×6
15168
−7616
7552

3808
3808
7616

2528
6

15168
7616
7552

Example :- Give the following system of eq^n.

$$x + y + z = 1.$$
$$x^2 + y^2 + z^2 = 2.$$
$$x^3 + y^3 + z^3 = 3.$$

Find the smallest the integer value of $n (>3)$ such that $x^n + y^n + z^n$ is an integer.

:- x, y, z are roots of some cubic eq^n.

$$a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0. \quad — (1)$$

Given that $S_1 = 1 = \frac{-a_1}{a_0}$

$$\boxed{a_1 = -a_0.}$$

$S_2 = 2$

$a_0 S_2 + a_1 S_1 + 2a_2 = 0.$

$2a_0 + (-a_0)(1) + 2a_2 = 0.$

$2a_0 - a_0 + 2a_2 = 0.$

$2a_2 = -a_0$

$$\boxed{a_2 = -\frac{1}{2} a_0.}$$

$S_3 = 3$

$a_0 S_3 + a_1 S_2 + a_2 S_1 + 3a_3 = 0.$

$a_0 (3) - a_0 (2) - \frac{a_0}{2}(1) + 3a_3 = 0.$

$$3a_0 - 2a_0 - \frac{a_0}{2} + 3a_3 = 0.$$

$$a_0 - \frac{a_0}{2} + 3a_3 = 0$$

$$\boxed{a_3 = -\frac{a_0}{6}}$$

Put $a_1 \& a_2 \& a_3$ in eq. 1

$$a_0x^3 - a_0x^2 - \frac{a_0}{2}x - \frac{a_0}{6} = 0.$$

$$\frac{6a_0x^3 - 6a_0x^2 - 3a_0x - a_0}{6} = 0.$$

$$a_0(6x^3 - 6x^2 - 3x - 1) = 0.$$

$$\underset{a_0}{6}x^3 - \underset{a_1}{6}x^2 - \underset{a_2}{3}x - \underset{a_3}{1} = 0.$$

$$a_0S_4 = -a_1S_3 - a_2S_2 - a_3S_1 - 4a_4$$

$$6S_4 = -(-6)(3) - (-3)(2) - (-1)(1)$$

$$6S_4 = +18 + 6 + 1.$$

$$S_4 = 25$$

$$S_4 = \frac{25}{6}.$$

$$a_0S_5 = -a_1S_4 - a_2S_3 - a_3S_2 - a_4S_1$$

$$6S_5 = -(-6)\frac{25}{6} - (-3)(3) - (-1)(2)$$

$$6S_5 = 25 + 9 + 2$$

DATE
PAGE

$\frac{32}{9}$

$S_5 = \frac{36}{6}$

$\boxed{S_5 = 6}$

$\therefore$ $n = 5$ is a smallest +ve integer value of n (>3).

Example :- If α, β, γ are roots of the eqn $x^3 + px + q = 0$. prove that $3(\alpha^2 + \beta^2 + \gamma^2)(\alpha^5 + \beta^5 + \gamma^5) = 5(\alpha^3 + \beta^3 + \gamma^3)(\alpha^4 + \beta^4 + \gamma^4)$.

$\therefore$ $x^3 + px + q = 0$. $\quad a_0x^3 + a_1x^2 + a_2x + a_3 = 0$

$S_1 = 0.$

$a_0S_2 = -a_1S_1 - 2a_2$
$= 0 - 2(p)$

$\boxed{S_2 = -2p}$

$a_0S_3 = -a_1S_2 - a_2S_1 - 3a_3.$
$= -3(q)$

$\boxed{S_3 = -3q}$

$a_0S_4 = -a_1S_3 - a_2S_2 - a_3S_1 -$
$= 0 - (p)(-2p) \ 0$

$\boxed{S_4 = 2p^2}$

$-30p^2q =$
$-10p^2q$

$a_0S_5 = -a_1S_4 - a_2S_3 - a_3S_2$
$= 0 - (p)(-3q) - (q)(-2p)$
$= 3pq + 2pq$

$\boxed{S_5 = 5pq}$

Scanned with CamScanner

Assume

We have to $3 S_2 S_5 = 5 S_3 S_4$

Put S_2 & S_5

$$3(-2p)(5pq) = 5(-3q)(2p^2)$$

$$-30p^2q = -30p^2q.$$

Hence Proved.

(2) If $\alpha, \beta, \gamma, \delta$ are roots of the eq^n

$\underset{a_0}{x^4} - \underset{a_1}{x^3} - \underset{a_2}{19x^2} + \underset{a_3}{49x} - \underset{a_4}{30} = 0$, Evaluate $\Sigma\alpha^5$ & $\Sigma\alpha^{-2}$

$\Sigma\alpha^5 = -2841$

$\Sigma\alpha^{-2} = \frac{1261}{900}$.

$S_1 = 1.$

$a_0S_2 + a_1S_1 + 2a_2 = 0.$

$S_2 + (-1)(1) + 2(-19) = 0.$

$S_2 - 1 - 38 = 0$

$\boxed{S_2 = 39}$

$a_0S_3 + a_1S_2 + a_2S_1 + 3a_3 = 0.$

$S_3 + (-1)(39) + (-19)(1) + 3(49) = 0.$

$S_3 - 39 - 19 + 147 = 0.$

$S_3 = -147 + 58$

$\boxed{S_3 = -89}$

$a_0S_4 + a_1S_3 + a_2S_2 + a_3S_1 + 4a_4 = 0.$

$S_4 + (-1)(-89) + (-19)(39) + (49)(1) + 4(-30) = 0$

$S_4 + 89 + 49 - 120 = 741$

$S_4 = 861 - 138$

$\boxed{S_4 = 723}$

DATE
PAGE

$a_0S_5 + a_1S_4 + a_2S_3 + a_3S_2 + a_4S_1 = 0$

$S_5 + (-1)(723) + (-19)(-89) + (49)(39) + (-30)(1) = 0.$

$S_5 - 723 - 30 + 1911 + 1691 = 0.$

$S_5 + 2849 = 0$

$\boxed{S_5 = -2849}$

for Σx^{-2} put $x = \frac{1}{y}$ in eqⁿ ①.

$\frac{1}{y^4} - \frac{1}{y^3} - \frac{19}{y^2} + \frac{49}{y} - 30 = 0.$

$\frac{1 - y - 19y^2 + 49y^3 - 30y^4}{y^4} = 0$

$\overset{a_0}{-30y^4} + \overset{a_1}{49y^3} - \overset{a_2}{19y^2} - \overset{a_3}{y} + \overset{a_4}{1} = 0.$

$a_0 = -30$ $\quad a_2 = -19$ $\quad a_4 = 1$

$a_1 = 49$ $\quad a_3 = -1$

$S_1 = \frac{-49}{-30} = \frac{49}{30}.$

$a_0S_2 + a_1S_1 + 2a_2 = 0.$

$(-30)S_2 + (49)\left(\frac{49}{30}\right) + 2(-19) = 0.$

$a_0S_2 + a_1S_1 + 2a_2 = 0$

$(-30)S_2 + (49)\left(\frac{49}{30}\right) + 2(-19) = 0$

$-30S_2 + \frac{98}{35} - 38 = 0$

$-900S_2 + 98 - 1140 = 0$

$-450S_2\ 549$

$-30S_2 + \frac{2401}{30} - 38 = 0.$ $\quad \frac{38}{5}$

$-900S_2 + 2401 - 1140 = 0$

$-900S_2 + 1261 = 0$

$S_2 = \frac{1261}{900}.$

* Newton's Method for finding sum of integer powers of roots.

Proof:- $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n = 0$

$f'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \dots + 2a_{n-2}x + a_{n-1}$ — ①

also if $f(x)$ has roots $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$.

$$f'(x) = \frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_2} + \dots + \frac{f(x)}{x-\alpha_n} \quad \text{— ②}$$

also $\frac{f(x)}{x-a}$

a	a_0	a_1	a_2	$\dots$
		aa_0	$a^2a_0+aa_1$	
	a_0	aa_0+a_1	$a^2a_0+a_1a+a_2$	

also $\frac{f(x)}{x-a} = a_0x^{n-1} + (aa_0+a_1)x^{n-2} + (a^2a_0+a_1a+a_2)x^{n-3} + \dots + (a^{n-2}a_0 + a^{n-1}a_1 + \dots + a_{n-2})x$

$+ (a^{n-1}a_0 + a^{n-2}a + \dots + a_{n-1})x^{n-n}$ — ③

from eq. ② & ③

$$\frac{f(x)}{x-\alpha_1} = a_0x^{n-1} + (\alpha_1a_0+a_1)x^{n-2} + (\alpha_1^2a_0 + a_1\alpha_1 + a_2)x^{n-3} + \dots$$

$$+ (\alpha_1^{n-2}a_0 + \alpha_1^{n-3}a_1 + \dots + \alpha_1^{n-2})x$$

$$+ (\alpha_1^{n-1}a_0 + \alpha_1^{n-2}a_1 + \dots + a_{n-1}).$$

DATE
PAGE

$$f'(x) = a_0x^{n-1} + (\alpha_1 a_0 + a_1)x^{n-2} + (\alpha_1^2 a_0 + a_1\alpha_1 + a_2)x^{n-3} + \dots$$
$$(\alpha_1^{n-2}a_0 + \alpha^{n-3}a_1 + \dots + a_{n-2})x$$
$$+ (\alpha_1^{n-1}a_0 + \alpha_1^{n-2}a_1 + \dots + a_{n-1})$$
$$+ a_0x^{n-1} + (\alpha_2 a_0 + a_1)x^{n-2} + (\alpha_2^2 a_0 + a_1\alpha_2 + a_2)x^{n-3} + \dots +$$
$$(\alpha_2^{n-2}a_0 + \alpha_2^{n-3}a_1 + \dots + \alpha_2^{n-2})x$$
$$-① \quad + (\alpha_2^{n-1}a_0 + \alpha_2^{n-2}a_1 + \dots + a_{n-1}) + \dots +$$
$$a_0x^{n-1} + (\alpha_n a_0 + a_1)x^{n-2} + (\alpha_n^2 a_0 + a_1\alpha_n + a_2)x^{n-3} + \dots +$$
$$(\alpha_n^{n-2}a_0 + \alpha_n^{n-3}a_1 + \dots + a_{n-2})x + (\alpha_n^{n-1}a_0 + \alpha_n^{n-2}a_1 + \dots + a_{n-1}).$$

$$f'(x) = na_0x^{n-1} + (a_0S_1 + na_1)x^{n-2} + (a_0S_2 + a_1S_1 + na_2)x^{n-3} + \dots + (a_0S_{n-2} + a_1 S_{n-1} + \dots + na_{n-2})x + (a_0S_{n-1} + a_1S_{n-2} + \dots + na_{n-1}).$$
— ④

from eqⁿ ① & ④.

$$(n-1)a_1 = a_0S_1 + na_1 \Rightarrow \cancel{n}a_1 - a_1 = a_0S_1 + \cancel{n}a_1$$
$$a_0S_1 + a_1 = 0$$
$$(n-2)a_2 = a_0S_2 + a_1S_1 + na_2$$
$$\cancel{n}a_2 - 2a_2 = a_0S_2 + a_1S_1 + \cancel{n}a_2$$
$$-2a_2 = a_0S_2 + a_1S_1$$
$$a_0S_2 + a_1S_1 + 2a_2 = 0.$$
⋮

Coeff. of x, $2a_{n-2} = a_0S_{n-2} + a_1S_{n-1} + \dots + na_{n-2}$

$$\Rightarrow a_0S_{n-2} + a_1S_{n-3} + \dots + (n-2)a_{n-2} = 0.$$

Constant term

$$a_{n-1} = a_0S_{n-1} + a_1S_{n-2} + \dots + na_{n-1}.$$
$$\Rightarrow a_0S_{n-1} + a_1S_{n-2} + \dots + (n-1)a_{n-1} = 0.$$

DATE
PAGE

$\Rightarrow$ We can find S_k when $k<n$.

If $k \geq n$ Multiply $f(x)$ by x^{k-n}

$\Rightarrow a_0x^k + a_1x^{k-1} + a_2x^{k-2} + \dots + a_{n-1}x^{k-n+1} + a_nx^{k-n} = 0$ — (5)

$\alpha_1, \alpha_2, \dots \alpha_n$ are roots of (5)

also

$\Rightarrow a_0\alpha_1^k + a_1\alpha_1^{k-1} + a_2\alpha_1^{k-2} + \dots + a_{n-1}\alpha_1^{k-n+1} + a_n\alpha_1^{k-n} = 0$

|||ly, $\Rightarrow a_0\alpha_2^k + a_1\alpha_2^{k-1} + a_2\alpha_2^{k-2} + \dots + a_{n-1}\alpha_2^{k-n+1} + a_n\alpha_2^{k-n} = 0$

$\vdots$

$\Rightarrow a_0\alpha_n^k + a_1\alpha_n^{k-1} + a_2\alpha_n^{k-2} + \dots + a_{n-1}\alpha_n^{k-n+1} + a_n\alpha_n^{k-n} = 0$

Adding all above eqⁿ

$\Rightarrow a_0S_k + a_1S_{k-1} + a_2S_{k-2} + \dots + a_{n-1}S_{k-n+1} + a_nS_{k-n} = 0$ — (6)

To find S_n, Put $k=n$.

$\Rightarrow a_0S_n + a_1S_{n-1} + a_2S_{n-2} + \dots + a_{n-1}S_1 + n a_n = 0$

To find S_{n+1}, Put $k=n+1$

$a_0S_{n+1} + a_1S_n + a_2S_{n-1} + \dots + a_{n-1}S_2 + a_nS_1 + (n+1)a_{n+1}$

where $a_{n+1}=0$

and so on...

To find S_{-k} where $k = 1, 2, 3, \ldots$, transform the given eq^n^ by $x \to \frac{1}{x}$ & find S_k in transformed eq^n^. It will be S_{-k} of given eq^n^.

Limits of Roots.

Superior Limit:-

* Superior Limit of +ve Roots:- It is any number greater than all positive roots.

OR

Any Greater No. than greatest of Positive root.

* Superior Limit of -ve Roots:- It is any greater number than largest of -ve roots.

Inferior Limit:-

* Inferior Limit of +ve Roots:- It is any number lesser (smaller) than all positive roots.

OR

(Any Smaller no. than smallest of +ve roots)

* Inferior Limit of -ve Roots:- It is any smaller no. than smallest of -ve roots.

Result:-

Superior limit of +ve roots.

For Polynomial.

$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1}x + p_n = 0.$

If. (i) $-p_r x^{n-r}$ is the first negative term.

(ii) p_k is the largest coeff. in -ve terms then superior limit is $(p_k)^{1/r} + 1$

[Here it means p_k is the largest coeff. in -ve terms without considering -ve sign.]

DATE
PAGE

Example :- $x^4 - 5x^3 + 40x^2 - 8x + 25 = 0.$

Compare with

$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n = 0.$

$n = 4.$

Step 1 :- Observe 1st negative term in given eqn.

$-5x^3 = p_1 x^{4-1}$

from this.

we see $r = 1.$

Step 2 :- Observe largest coeff. in -ve terms without -ve sign is 8.

$\therefore p_k = 8.$

Now

$(p_k)^{1/r} + 1 = (8)^{1/1} + 1$

$= 9.$

Result 2 :-

For Polynomial

$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_r x^{n-r} + \dots + a_n = 0,$

take negative coefficients & then without considering their -ve sign, divide each of them by sum of positive coeff. preceding that -ve coeff., then adding 1 in each quotient, superior limit of +ve roots is no. greater than these quotients.

Example :- $x^4 - 5x^3 + 40x^2 - 8x + 25 = 0.$

Observe -ve terms i.e. 5, 8.

$\frac{5}{1} + 1 \quad , \frac{8}{1+40} + 1$

$= \underline{6} \quad , 1.$

DATE
PAGE

Example $x^5+3x^4+x^3-8x^2-51x+18=0$.

1st Result :-

1st neg. term $= -8x^2 = p_3 x^{5-3}$

$r=3$.

$\boxed{p_n = 51}$

$(p_n)^{1/r}+1$

$(51)^{1/3}+1 \Rightarrow 4+1$

Sup. limit $= 5$.

2nd Result :-

$8, 51$.

$\frac{8}{5}+1, \frac{51}{5}+1$

Sup. limit $= 2.6, 11.2$. (max.)

Hence, the superior limit is 5.

(2) $x^7+4x^6-3x^5+5x^4-9x^3-11x^2+6x-8=0$

Result 1st :-

$-3x^5 = -p_2 x^{7-2}$

$\boxed{r=2}$

$(+11)^{1/2}+1 \Rightarrow 4+1$

$=5$.

Scanned with CamScanner

Result 2nd,

$$\frac{3}{5}+1, \frac{9}{6}+1, \frac{11}{10}+1, \frac{8}{16}+1$$

$$= \frac{8}{5}, \frac{15}{6}, \frac{21}{10}, \frac{24}{16}$$

$= 1.6, 1.9, 2.1, 1.5.$

Sup. limit $= 2.1.$

Hence, the superior limit is 2.1.

Example:-

(1) $x^8 + 20x^7 + 4x^6 - 11x^5 - 120x^4 + 13x - 25 = 0.$ (6)

(2) $4x^5 - 8x^4 + 22x^3 + 98x^2 - 73x + 5 = 0.$ (3)

(1) $\underset{p_0}{x^8} + \underset{p_1}{20x^7} + \underset{p_2}{4x^6} \underset{p_3}{- 11x^5} - \underset{p_4}{120x^4} + \underset{p_7}{13x} - \underset{p_8}{25} = 0.$

1st Result:-

1st negative term $= -11x^5 = -p_3 x^{8-5}$

Here, $r = 3$

2nd Step. $(120)^{1/3} + 1 = 6.$

2nd Result:-

$$\frac{11}{25}+1, \frac{120}{25}+1, \frac{25}{25+13}+1$$

$$= \frac{36}{25}, \frac{145}{25}, \frac{63}{38}$$

$= 1.6, 5.9, 1.6$

In this case 5.9 is a superior limit or we can say it 6.

(a) Hence, superior limit is 6.

(2) $\underset{p_0}{4x^5} - \underset{p_1}{8x^4} + \underset{p_2}{22x^3} + \underset{p_3}{98x^2} - \underset{p_4}{73x} + \underset{p_5}{9} = 0$

1st Result.

1st step :- $-8x^4 = -p_1 x^{5-1}$

$\therefore x = 1.$

2nd Step :- $(73)^1 + 1 = 9 + 1$

$= 10.$

2nd Result :-

$= \frac{8}{4} + 1 \ , \ \frac{73}{124} + 1$

$= 3, \ \frac{197}{124}$

$= 3, 1.4.$

In 2nd Result. Superior limit is 3.

Overall, The Superior limit is 3.

(1) With Grouping Method.

$(x^8 - 11x^5) + (20x^7 - 120x^4) + (4x^6 - 25) + 13x = 0$

$\cancel{x^5(x^3-11) + 12x^4(x^3 - 60x)}$

$x^5(x^3 - 11) + 20x^4(x^3 - 6) + 4x^6 + 13x - 25 = 0$

If $x = 3$

All terms becomes +ve.

$\therefore$ 3 is a superior limit.

DATE
PAGE

3rd Result

(Grouping Method)

$x^5+3x^4+x^3-8x^2-51x+18=0.$

$=(x^5-8x^2)+(3x^4-51x)+(x^3+18)=0.$

$=x^2(x^3-8)+x(3x^3-51)+x^3+18=0.$

Put value of x so that -ve term converted into +ve. 32

98 -

By Putting 2 it become negative.

$\therefore$ We have to put 3.

$x=3.$

become +ve all the terms.

$\therefore$ Hence, Superior limit is 3.

Example: $x^7+4x^6-3x^5+5x^4-9x^3-11x^2+6x-8=0.$

$(x^7-3x^5)+(4x^6-9x^3)+(5x^4-11x^2)+(6x-8)=0.$

$=x^5(x^2-3)+x^3(4x^3-9)+x^2(5x^2-11)+(6x-8)=0$

If $x=2$

All terms becomes +ve.

$\therefore$ 2 is a superior limit.

(2) $4x^5 - 8x^4 + 22x^3 + 98x^2 - 73x + 5 = 0$.

$(4x^5 - 8x^4) + (22x^3 - 73x) + (98x^2 + 5)$.

$4x^4(x-2) + x(22x^2 - 73) + (98x^2 + 5) = 0$

If $x = 2$

All terms are +ve.

$\therefore$ 2 is superior limit.

29/April.

Example $6x^5 - 7x^4 - 10x^3 - 23x^2 - 90x - 317 = 0$.

$(x^5 - 7x^4) + (x^5 - 10x^3) + (x^5 - 23x^2) + (x^5 - 90x) + (x^5 - 317) = 0$

$x^4(x-7) + x^3(x^2 - 10) + x^2(x^3 - 23) + x(x^4 - 90) + (x^5 - 317) = 0$

if $x = 7$

All the terms become +ve.

$\therefore$ 7 is a superior limit of +ve roots.

(2) $x^4 - x^3 - 2x^2 - 4x - 24 = 0$. — (1)

Multiply eq. (1) with 4

$4x^4 - 4x^3 - 8x^2 - 16x - 96 = 0$.

$(x^4 - 4x^3) + (x^4 - 8x^2) + (x^4 - 16x) + (x^4 - 96) = 0$

$x^3(x-4) + x^2(x^2 - 8) + x(x^3 - 16) + (x^4 - 96) = 0$.

if $x = 4$,

All the terms become +ve.

$\therefore$ 4 is a superior limit of +ve roots.

DATE
PAGE

(3) Newton's Method :-

Newton's Method to find Superior limit of +ve roots.

Eg (1) $x^4 - 2x^3 - 3x^2 - 15x - 3 = 0$

$f'(x) = 4x^3 - 6x^2 - 6x - 15 = 2x^2(x-3) - x(x^2-6) + (x^2-15)$

$f''(x) = 12x^2 - 12x - 6 = 6(2x^2 - 2x - 1)$

$f'''(x) = 24x - 12 = 12(2x-1)$

$f^{iv}(x) = 24.$

$\therefore$ 4 is a superior limit.

Eg (2) $2x^5 - 7x^4 - 5x^3 + 6x^2 + 3x + 1 = 0$

$f_1(x) = 10x^4 - 28x^3 - 15x^2 + 12x + 3 \Rightarrow x^3(5x-14) + (12x - 15x^2) + 3$

$f_2(x) = 40x^3 - 84x^2 - 30x + 12$

$f_3(x) = 120x^2 - 168x - 30 = 2(60x^2 - 84x - 15) \Rightarrow 6(20x^2 - 28x - 3)$

$f_4(x) = 240x - 168 = 4(60x - 42) \Rightarrow 24(10x - 7)$

$f_5(x) = 240.$

$\therefore$ 4 is a Superior limit of +ve roots.

$32 - 24 - 18$

$108 - 54 - 18 - 15$

$4x^2(20x - 14)$

$4x^2 \times 2(10x - 7)$

$(120x^2 - 168x - 30)$

$2(60x^2 - 84x - 15)$

900

$6(20x^2 - 28x - 3)$

$3x(4 - 5x)$

$4x^2(10 - 21x)$

Shortcut.

$$f(x) = \left(\left(\left((2x-7)x-5\right)x+6\right)x+3\right)x$$

* Inferior limit of +ve Roots.

Observation.

1 Step. If $f(x)$ is given, find (Sup. limit of +ve roots) of $f\left(\frac{1}{x}\right)$ = a

Then $\frac{1}{a}$ will be inf limit of +ve roots of given eq^n.

$\boxed{\text{let } f(x) \text{ has roots } -100, -4, 8, 5, 7 \\ \text{roots of } f\left(\frac{1}{x}\right) \text{ are } -\frac{1}{100}, -\frac{1}{4}, \frac{1}{8}, \frac{1}{5}, \frac{1}{7}}$

* Inferior limit of -ve roots.

roots $f(-x)$ are $100, 4, -8, -5, 7$.

If $f(x)$ is given, find (sup. limit of +ve roots) of $f(-x)$ = b. If it is b. Then $-b$ is inf. limit of -ve roots of $f(x)$.

* Superior limit of -ve roots.

Find (inf. limit of -ve roots) of $f\left(\frac{1}{x}\right)$ = c. If it is c, this is sup. limit of -ve roots.

Or

Given $f(x)$. Find (sup. limit of +ve roots) of $f\left(\frac{-1}{x}\right)$ = d.

if it is d, then $\frac{-1}{d}$ is sup. limit of -ve roots.

(1) To find superior limit of +ve roots — Some Methods

(2) To find Inferior limit of +ve roots. — taking

Find superior limit of +ve roots of $f(\frac{1}{x})$. Then it's reciprocal, we get inferior limit of +ve roots.

(3) Negative Roots.

(i) Take $f(-x)$ instead of $f(x)$. — [Inferior limit of +ve roots of $f(x)$]

(ii) Find sup. limit of +ve roots of $f(-x)$ & inf. limit of +ve roots of $f(x)$.

(iii) If these values are h & h' respectively. then, $-h$ & $-h'$ are inferior & superior limit of -ve roots of $f(x)$ resp.

Example: Show that real roots of eq^n

$$x^4 - 44x^2 + 112x - 384 = 0$$

lie b/w -12 & 7.

$x^2(x^2-44) + (112x - 384) = 0.$

$+ 4\,(28x - 96)$

$+ 8(14x - 48)$

$x^2(x^2-44) + 16(7x - 24) = 0.$

If $x = 7$,

then all terms become +ve.

$\therefore$ 7 is a superior limit of +ve roots.

$f(-x) = x^4 - 44x^2 - 112x - 384 = 0.$

Multiply by (2)

$(x^4 - 132x^2) + (x^4 - 336x) + (x^4 - 1152) = 0$

$x^2(x^2 - 132) + (x^4 - 336x) + (x^4 - 1152) = 0$

If $x = 12$
then all terms become +ve.
$\therefore$ 12 is a superior limit of +ve roots of $f(-x)$
$\Rightarrow x = -12$ is inf. limit of −ve roots of $f(x)$

(2) Find upper and lower limits of +ve roots of eq:
$2x^5 - 7x^4 - 5x^3 + 6x^2 + 3x - 10 = 0$

$x^4(2x - 7) + x^2(6 - 5x) + (3x - 10) = 0.$

$(x^5 - 7x^4) + (x^5 - 5x^3) + (6x^2 - 10) + 3x = 0.$
$x = 7$ is sup. limit of +ve roots.

1st Method :-
(1) $-7x^4 = -p\,x^{5-r}$
$\boxed{r = 1}$

(2) $(10)^{1/1} + 1$
$= 11.$

2nd Method :-
$\frac{7}{2} + 1, \quad \frac{5}{2} + 1, \quad \frac{10}{11} + 1$

$\frac{9}{2}, \quad \frac{7}{2}, \quad \frac{10 + 11}{11}$

$\frac{9}{2}, \frac{7}{2}, \frac{21}{11}$

$= 4.5, 3.5, 1.9.$
Max. value is 4.5.

(1-20)
$256 \times (?) + 16(-14) + 2$
$256 - 2?2$

$256 + 16(?)$
$258 - 22?$

By Newton's Method.

For inferior limit of +ve roots.

$$\frac{2}{x^5} - \frac{7}{x^4} - \frac{5}{x^3} + \frac{6}{x^2} + \frac{3}{x} - 10 = 0$$

$$2 - 7x - 5x^2 + 6x^3 + 3x^4 - 10x^5 = 0.$$

$$10x^5 - 3x^4 - 6x^3 + 5x^2 + 7x - 2 = 0.$$

~~$(10x^5 - 3x^4) + x^2(5 - 6x)$ [illegible] $(7x - 2) = 0.$~~

~~[illegible]~~

$$(5x^5 - 3x^4) + (5x^5 - 6x^3) + (5x^2 - 2) + 7x = 0.$$

$$x^4(5x - 3) + x^3(5x^2 - 6) + (5x^2 - 2) + 7x = 0.$$

If $x = 2$

then all terms become +ve

$\therefore$ 2 is superior limit of +ve roots of $f\left(\frac{1}{x}\right)$.

$x = \frac{1}{2}$ is a ~~superior~~ inferior limit of +ve roots

Example :- Find sup. limit & inf. limit of the roots of
$x^4 - 3x^3 - 2x^2 + 7x + 3 = 0.$

$f(-x) = x^4 + 3x^3 - 2x^2 - 7x + 3 = 0.$
$\quad\;\; p_0 \quad p_1 \quad p_2 \quad p_3 \quad p_4$

1st Method.

$-2x^2 = -p_2 x^{4-2}$

$\therefore r = 2.$

~~2nd method~~ $(p_r)^{1/r} + 1 = (7)^{1/2} + 1$

$= 3+1$

$= 4.$

$\therefore$ 4 is a superior limit of 1st Method.

2nd Method :-

$\frac{2}{4} + 1 \quad , \quad \frac{7}{4} + 1$

$= \frac{1+2}{2} \quad , \quad \frac{7+4}{4}$

$= \frac{3}{2} , \frac{11}{4}.$

$= 1.5 , 2.7.$

2.7 is a superior limit of 2nd Method.

3rd Method.

$x^2(x^2 - 2) + x(3x^2 - 7) + 3 = 0.$

if $x = 2$

then all terms become +ve

$\therefore$ 2 is a superior limit of 3rd Method.

Scanned with CamScanner

Newton's Method.

$$f(-x) = x^4 + 3x^3 - 2x^2 - 7x + 3 = 0$$

$$f_1(-x) = 4x^3 + 9x^2 - 4x - 7. \Rightarrow 4x(x^2-1) + (9x^2-7)$$

$$f_2(-x) = 12x^2 + 18x - 4 \Rightarrow (12x^2-4) + 18x \Rightarrow 4(3x^2-1) + 18x.$$

$$f_3(-x) = 24x + 18 = 3(8x+6). = 6(4x+3).$$

$$f_4(-x) = 24.$$

$x = \frac{3}{7}$

$x=3$ is a superior limit of this method.

$1+3-2-7+3$

∴ $x=2$ is a superior limit of +ve roots of $f(-x)$

it means.

$x=-2$ is a inferior limit of -ve roots of $f(x)$.

Superior limit of negative roots.

$$f(-x) = x^4 + 3x^3 - 2x^2 - 7x + 3 = 0$$

$$f\left(\frac{1}{x}\right) = \frac{1}{x^4} + \frac{3}{x^3} - \frac{2}{x^2} - \frac{7}{x} + 3.$$

$$= 1 + 3x - 2x^2 - 7x^3 + 3x^4.$$

$$= 3x^4 - 7x^3 - 2x^2 + 3x + 1$$

1st Method

$$-7x^3 = -p_r x^{4-r}$$

$$\therefore \boxed{r=1}$$

$$(7)^{1/1} + 1$$

$$= 8.$$

∴ 8 is a superior limit of +ve roots $f\left(\frac{1}{x}\right)$ in 1st Method.

2nd Method.

$\frac{7}{3}+1, \quad \frac{2}{3}+1$

$=\frac{10}{3}, \frac{5}{3}.$

$= 3.3, 1.6.$

$\therefore$ 3.3 is a superior limit of +ve roots of $f(\frac{1}{x})$ in 2nd Method

3rd Method.

$(x^4-7x^3)+(2x^4-2x^2)+3x+1$

$x^3(x-7)+2x^2(x^2-1)+3x+1$

If $x=7$

then all terms become +ve

$\therefore x=7$ is a superior limit of +ve roots of $f(\frac{1}{x})$ in 3rd Method.

4th Method:-

$3x^2(2x-7)+2x(3x-2)+3.$

$f(x)=3x^4-7x^3-2x^2+3x+1$

$f_1(x)=12x^3-21x^2-4x+3 \Rightarrow (6x^3-21x^2)+(6x^3-4x)+3 \Rightarrow$

$f_2(x)=36x^2-42x-4 \Rightarrow 6x(6x-7)-4.$

$f_3(x)=72x-42 \quad = 6(12x-7).$

$f_4(x)=72$

$x=4$ is a superior limit of +ve roots of $f(\frac{1}{x})$ in

4th Method.

Overall, 4 is a superior limit of +ve roots of $f(\frac{1}{x})$.

$\therefore \frac{1}{4}$ is a inferior limit of +ve roots of $f(-x)$.

it means

$x = -\frac{1}{a}$ is a superior limit of -ve roots of $f(x)$.

Unit :- 4.

Discrates Rule :-

No. of +ve real roots $\leq$ no. of variation of signs in $f(x)$

No of -ve " " $\leq$ " " " " $f(-x)$

Sturm's Theorem :-

Statement :- Let any two real quantities a & b be substituted for x in series of n+1 functions $f(x)$, $f_1(x)$, $f_2(x)$, ..., $f_n(x)$ consisting of the given polynomial $f(x)$, its first derivative $f_1(x)$ & the successive remainders (with their sign changed) in the process of finding GCD of $f(x)$ & $f_1(x)$. Then the difference between number of changes of signs in the series when 'a' is substituted for x & when 'b' is substituted for x expresses exactly the no. of real roots of eq: $f(x) = 0$ b/w a & b.

How to find GCD.

Dividend = Divisor . Quotient + remainder.

we have 20, 36.

$$\begin{array}{l} 20 \overline{)36}\,(1 \\ \quad 20 \\ \quad 16 \overline{)20}\,(1 \\ \qquad 16 \\ \qquad 4 \overline{)16}\,(4 \\ \qquad\quad 16 \\ \qquad\quad 0 \end{array}$$

$36 = 20.1 + 16$

$20 = 16.1 + 4$

$16 = 4.4 + 0.$

DATE
PAGE

Example:- Find no. & situation of the real roots of
$f(x) = x^3 - 2x - 5$.

Sol: $f(x) = x^3 - 2x - 5$
$\therefore$ $f_1(x) = 3x^2 - 2$

$$\boxed{\begin{array}{l} 3x^2 - 2 \,\big)\, x^3 - 2x - 5 \,\big(\, x \\ \qquad\quad 3x^3 - 2x \\ \qquad\quad - \;\; + \\ \hline \qquad\qquad -2x^3 - 5 \,\big)\, 3x^2 - 2 \,\big(\end{array}}$$

$-2x^3 \times \frac{1}{x} - \frac{5}{x}$

$$\begin{array}{l} 3x^2 - 2 \,\big)\, x^3 - 2x - 5 \,\big(\, \frac{1}{3}x \\ \qquad\quad x^3 - \frac{2x}{3} \\ \qquad\quad - \;\; + \\ \hline \qquad\quad -\frac{4}{3}x - 5 \end{array}$$

$\frac{2}{3} - 2$

$\frac{2-3}{3}$

$-\frac{1}{3x}$

$\therefore f_2(x) = \frac{4}{3}x + 5.$

$\frac{2}{3} - 2$

$\frac{2-6}{3}$

$f_2(x) = \frac{1}{3}(4x + 15).$

$$\begin{array}{l} \frac{1}{3}(4x+15) \,\big)\, 3x^2 - 2 \,\big(\, \frac{9}{4}x - \frac{135}{16} \\ \qquad\qquad 3x^2 + \frac{45}{4}x \\ \qquad\qquad - \quad\;\; - \\ \hline \qquad\qquad -2 - \frac{45}{4}x. \\ \qquad\qquad -\frac{675}{16} - \frac{45}{4}x \\ \qquad\qquad + \qquad + \\ \hline \qquad\qquad \frac{675 - 32}{16} = \frac{643}{16}. \end{array}$$

$\frac{4}{3}x \cdot \frac{9}{4}x$

$\frac{3}{3} \quad \frac{9}{4}$

$f_3(x) = \frac{-643}{16}$

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	Variation
$x=\infty$	+	+	+	-	1
$x=0$	-	-	+	-	2
$x=-\infty$	-	+	-	-	2

1) No. of +ve real roots b/w 0 & ∞
= No. of variation of signs at 0 - No. of var. of sign at ∞
= 2 - 1
= 1.

2) No. of -ve real roots b/w $-\infty$, 0.
= No. of variation of signs at $-\infty$ - No. of variation of sign at 0.
= 2 - 2
= 0.

$\therefore$ To find situation of real roots.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	Variation
$x=1$	-	+	+	-	2.
$x=2$	-	+	+	-	2.
$x=3$	+	+.	+	-	1

Real Root lie b/w. 2 & 3.
= No. of ~~real~~ ~~root~~ variation of sign at 2 - No. of variation of sign at 3
= 2 - 1 = 1.

DATE
PAGE

Example ② $x^3 - 7x + 7$

$f(x) = x^3 - 7x + 7$ ①

Multiply with 3.

$f(x) = 3(x^3 - 7x + 7)$.

Derivative of ①

$f_1(x) = 3x^2 - 7$

$$3x^2 - 7 \,\big)\, 3x^3 - 21x + 21 \,\big(\, x$$

$$\cancel{3x^3} - 7x$$

$$- \quad +$$

$$-14x + 21.$$

$f_2(x) = 14x - 21. \Rightarrow 7(2x - 3).$

$$14x - 21 \,\big)\, 3x^2 - 0x - 7 \,\big(\, \frac{3}{14}x + \frac{9}{28}$$

$$\cancel{3x^2} - \frac{9}{2}x.$$

$$- \quad +$$

$$\frac{9}{2}x - 7.$$

$$\cancel{\frac{9}{2}x} - \frac{27}{4}$$

$$- \quad +$$

$$\frac{27}{4} - 7 \Rightarrow \frac{27 - 28}{4} = -\frac{1}{4}$$

$f_3(x) = \frac{1}{4}.$

DATE
PAGE

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	Variation
$x=\infty$	+	+	+	+	0
$x=0$	+	−	−	+	2
$x=-\infty$	−	+	−	+	3.
$x=-3$	+	+	−	+	1
$x=-4$	−	+	−	+	3.

⇒ No. of +ve real roots b/w $0, \infty$

= No. of variation of signs at 0 − No. of variation of signs at ∞

= 2 − 0

= 2.

⇒ No. of −ve real roots b/w $-\infty, 0$.

= 3 − 2

= 1.

Now. Situation of roots.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	Variation
$x=1$	+	−	+	+	2
$x=2$	+	+	+	+	0.
$x=-1$	+				

$f\left(\frac{-1}{x}\right) =$

$= 1 - 3x - 2x^2 + 7x^3 + 3x^4$

$3x^4 + 7x^3 - 2x^2 - 3x + 1$

$f(-x)$

$= 3x^4 - 7x^3 - 2x^2 + 3x + 1$ $f\left(\frac{1}{x}\right)$

108
×30
0000
324×0

$32 - 24 - 12 + 6$

162
×3
486

$256 - 128 - 48 + 40 - 4$

$128 - 8 - 4$.

128
64