Algebra UNIT-1

Complex No. or De-Moivre's thm

Definition : Any number of the form $a+ib$, where $a,b \in R$

De-moivre's theorem

If $\cos\theta + i\sin\theta = e^{i\theta} = cis\theta$

If n is any integer then, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$(\cos\theta + i\sin\theta)^{-3} = \cos(-3\theta) + i\sin(-3\theta)$
$= \cos 3\theta - i\sin 3\theta$

* $(\cos\theta + i\sin\theta)^{1/q}$, $q \neq 0$
$(\cos\frac{1}{q}\theta) + i\sin\frac{1}{q}\theta)$

* $(\cos\theta + i\sin\theta)^{p/q}$ $q \neq 0$
$\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$

Basic concepts

1. If $z = x+y$ is a complex no. then we can write $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2+y^2}$ and θ is the amplitude, $\theta = \tan^{-1}\frac{y}{x}$ and we always take place principal value of a $r = \sqrt{x^2+y^2}$ is called modulus of complex no. θ is called amplitude or argument.

2. Principal value of a $-\pi < \theta \leq \pi$

3. $z = r(\cos\theta + i\sin\theta)$ is also called modulus amplitude form.

4. $\cos\theta + i\sin\theta$ is generally written as $\text{cis}\,\theta$
5. $\sin(\theta + 2\pi) = \sin\theta$
$\cos(\theta + 2\pi) = \cos\theta$
$\tan(\theta + \pi) = \tan\theta$

So, sin and cos are periodic functions with period 2π and tan is periodic with period π.

Let n be any integer, then

$\sin(\theta + 2n\pi)$

$n = 2$

$\sin(\theta + 4\pi) =$
$\sin(\theta + 2\pi + 2\pi)$
$\sin 2\pi \Rightarrow \sin\theta$

$n = -2$

$\sin(\theta - 4\pi)$
$\sin(\theta - 4\pi + 2\pi)$
$\sin 2\pi \Rightarrow \sin\theta$

$\cos(\theta + 2n\pi)$

$n = 2$

llly $\cos(\theta + 4\pi) \Rightarrow \cos(\theta + 2\pi + 2\pi)$
$\cos 2\pi$
$\cos\theta$

Sum (1)

Given :- $a = cis\,\alpha$

$b = cis\,\beta$

$c = cis\,\gamma$

and $a+b+c=0$

Prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

Hint :- Prove that $\frac{1}{cis\theta} = (cis\,-\theta)$

$$\frac{1}{cis\theta} \times \frac{cis\theta}{cis\theta} \Rightarrow \frac{cis\theta}{cis}$$

$$\frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i^2\sin^2\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \cos\theta - i\sin\theta$$

$$L.H.S = \frac{1}{cis\alpha} + \frac{1}{cis\beta} + \frac{1}{cis\gamma}$$

$$= \frac{1}{\cos\alpha + i\sin\alpha} + \frac{1}{\cos\beta + i\sin\beta} + \frac{1}{\cos\gamma + i\sin\gamma}$$

$$= \cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma$$

$$= \cos\alpha + \cos\beta + \cos\gamma - i\sin\alpha - i\sin\beta - i\sin\gamma$$

$$= \cos\alpha + \cos\beta + \cos\gamma - i[\sin\alpha + \sin\beta + \sin\gamma)$$

$a+b+c=0$

$cis\alpha + cis\beta + cis\gamma = 0$

$(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0+0i$

$= 0$

~~Sum Given $a+b+c=0$~~

~~$cis\alpha + cis\beta + cis\gamma = 0$~~

~~$(\cos\alpha + i\sin\alpha) + (\cos\beta + \sin\beta) + \cos\gamma + i\sin$~~

Sum (2) Prove that

$cis\theta_1 \, cis\theta_2 = cis(\theta_1+\theta_2)$

L.H.S $= cis\theta_1 \, cis\theta_2 = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1(\cos\theta_2 + i\sin\theta_2) + i\sin\theta_1(\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1 \cdot \cos\theta_2 + \cos\theta_1 i\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2$

$= \cos\theta_1\cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$

$= \cos(\theta_1+\theta_2) + i(\sin(\theta_1+\theta_2))$

$= cis(\theta_1+\theta_2)$

R.H.S

Sum (3) Prove that

$\frac{cis\theta_1}{cis\theta_2} = cis(\theta_1-\theta_2)$

Ans $(cis\theta_1)\, cis(-\theta_2) = cis(\theta_1-\theta_2)$

L.H.S = R.H.S

De-Movire thm.

Case I: $cis\,\theta_1\theta_2 = cis(\theta_1+\theta_2)$

$cis\,\theta_1\, cis\,\theta_2\, cis\,\theta_3 = cis(\theta_1+\theta_2)\,cis\,\theta_3$

$= cis(\theta_1+\theta_2+\theta_3)$

$cis\,\theta_1\, cis\,\theta_2\, cis\,\theta_3 \ldots cis\,\theta_n = cis(\theta_1+\theta_2+\theta_3 \ldots \theta_n)$

Making $\theta_1=\theta_2=\theta_3=\ldots=\theta_n=\theta$

L.H.S $cis\,\theta\, cis\,\theta\, cis\,\theta = cis\,\theta$

$= (cis\,\theta)^n = (\cos\theta + i\sin\theta)^n$

R.H.S $= cis(\underbrace{\theta+\theta+\theta \ldots +\theta}_{n\text{ times}}) = cis\, n\theta = \cos n\theta + i\sin n\theta$

This proves De-moivre thm. for +ve integers

Case II: Suppose n is a negative integer.

$-n$ is +ve

take $m=-n$

$n=-m$, where m is +ve integer.

To Prove: $(cis\,\theta)^n = cis\, n\theta$

L.H.S $= (cis\,\theta)^n \quad (cis\,\theta)^{-m} \Rightarrow \frac{1}{(cis\,\theta)^m} = \frac{1}{}$

$= \frac{1}{(cis\,\theta)^m} = \frac{1}{cis\, m\theta}$ (By De-moivre thm for +ve integer)

$= cis(-m\theta)$ $\quad -m=n$

$= cis\, n\theta$

R.H.S

This proves De-moivre thm for -ve integer.

Case III: If $n=0$

L.H.S $= (cis\,\theta)^0 = 1$

R.H.S: $cis\,0\theta = cis\,0$

$= \cos 0 + i\sin 0$

$= 1$

case IV

If n is a fraction or rational, then one of the value of $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$

Proof:- consider $n = \frac{p}{q}$, $q \neq 0$, $(p, q) = 1$ &

p, q are integer

$\Rightarrow \left(\cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta\right)^q = \cos\frac{1}{\cancel{q}}\times\cancel{q}\theta + i\sin\frac{1}{\cancel{q}}\times\cancel{q}\theta$

$\Rightarrow$ " $= \cos\theta + i\sin\theta$

$\Rightarrow \cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta$ is one of value of $(\cos\theta + i\sin\theta)^{1/q}$

$\Rightarrow \left(\cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta\right)^p$ is one of the values of $\left((\cos\theta + i\sin\theta)^{1/q}\right)^p$

$\Rightarrow \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)$ is one of the value of $(\cos\theta + i\sin\theta)^{p/q}$

$\Rightarrow (\cos n\theta + i\sin n\theta)$ is one of the value of $(\cos\theta + i\sin\theta)^n$

$\cos\theta + i\sin\theta = cis\,\theta$

$\cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$
$= cis(-\theta)$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$(\cos\theta - i\sin\theta)^n = ?$

L.H.S $cis(-\theta)^n = cis(-n\theta) = \cos(-n\theta) + i\sin(-n\theta)$
$= \cos n\theta - i\sin n\theta$

$$\frac{1}{\text{cis}\,\theta} = \text{cis}(-\theta)$$

$$\frac{1}{\cos\theta - i\sin\theta} = \frac{1}{\text{cis}(-\theta)} = \text{cis}\,\theta$$

$(\sin\theta + i\cos\theta)^n$

$= \{i(-i\sin\theta + \cos\theta)\}^n$

$= i^n\,\text{cis}(-n\theta)$

$= i^n(\cos n\theta - i\sin n\theta)$

$= i^n\cos n\theta - i^{n+1}\sin n\theta$

$\sin\theta + i\cos\theta$:-
$\{\cos(\frac{\pi}{2}-\theta) + i\sin(\frac{\pi}{2}-\theta)\}^n$

$\cos n(\frac{\pi}{2}-\theta) + i\sin n(\frac{\pi}{2}-\theta)$

$$\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$$

$$= \frac{\cos 4\theta + i\sin 4\theta}{\{\cos(\frac{\pi}{2}-\theta) + i\sin(\frac{\pi}{2}-\theta)\}^4}$$

$$= \frac{\text{cis}\,4\theta}{\text{cis}\,4(\frac{\pi}{2}-\theta)}$$

$$= \frac{\text{cis}\,4\theta}{\text{cis}\,2\pi - 4\theta} \Rightarrow \frac{\text{cis}\,4\theta}{\text{cis}(-4\theta)}$$

$\Rightarrow \text{cis}\,4\theta\,\text{cis}\,4\theta$

$= \text{cis}(4\theta + 4\theta) \Rightarrow \text{cis}(8\theta)$

* How $cis(2\pi - 4\theta) = cis(-4\theta)$?

$= \cancel{cis}\ \cos(2\pi - 4\theta) + i\sin(2\pi - 4\theta)$
$= \cos(4\theta) + i\sin(-4\theta)$
$= \cos(-4\theta)$

Sum of infinite terms of G.P with first term a and common ratio $r = \frac{a}{1-r}$

Exercise:- If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, prove that

$x_1 x_2 x_3 \ldots \infty = -1$

Sol:- $x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = cis\frac{\pi}{2}$

$x_2 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = cis\frac{\pi}{4}$

$x_3 = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8} = cis\frac{\pi}{8}$

L.H.S $= x_1 x_2 x_3 \ldots \infty$

$\cancel{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}$

$cis\frac{\pi}{2}\ cis\frac{\pi}{4}\ cis\frac{\pi}{8} \ldots \infty$

$cis\left[\frac{\pi}{2} \cdot \frac{\pi}{4} \cdot \frac{\pi}{8} \ldots \infty\right)$

$\frac{\frac{\pi}{4} \cdot 2}{\cancel{\pi}}$ or r

Here $a = \frac{\pi}{2}$ $r = \frac{\pi}{4} \times \frac{2}{\pi} = \frac{1}{2}$ By using G.P

$s = \frac{a}{1-r}$

$s = \frac{\frac{\pi}{2}}{1 - \frac{1}{2}} \Rightarrow s = \frac{\pi}{2} \cdot 2 = \pi$

$\cos\pi + i\sin\pi =$

$-1 + 0$

-1

L.H.S = R.H.S

Example Prove that general values of θ, which satisfies the equation

$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = 1$

is $\frac{4m\pi}{n(n+1)}$, where m is any integer.

Sol: L.H.S $cis(\theta + 2\theta + 3\theta + \ldots + n\theta)$

$cis(1+2+3+\ldots+n)\theta$

$cis \frac{n(n+1)\theta}{2} \Rightarrow \cos\frac{n(n+1)}{2}\theta + i\sin\frac{n(n+1)}{2}\theta$

R.H.S $= 1 = 1 + 0i$

$\Rightarrow \cos\frac{n(n+1)}{2}\theta = 1$ and $\sin\frac{n(n+1)}{2}\theta = 0$

$\Rightarrow \cos\frac{n(n+1)}{2}\theta = \cos 0 \quad \cos(0 + 2m\pi)$, where $m \in Z$

$\Rightarrow \frac{n(n+1)}{2} = 2m\pi, \; m \in Z$

$\Rightarrow \theta = \frac{4m\pi}{n(n+1)}, \; m \in Z$

Example:- If $z = \cos\theta + i\sin\theta$ prove that $\frac{z^{2n} - 1}{z^{2n} + 1} = i\tan n\theta$

Sol:- $\dfrac{z^{2n}-1}{z^{2n}+1} = i\tan n\theta$

$$\frac{(\cos\theta + i\sin\theta)^{2n} - 1}{(\cos\theta + i\sin\theta)^{2n} + 1}$$

$$\frac{\cos 2n\theta + i\sin 2n\theta - 1}{\cos 2n\theta + i\sin 2n\theta + 1}$$

$$\frac{i\sin 2n\theta - (1-\cos 2n\theta)}{i\sin 2n\theta + (1+\cos 2n\theta)}$$

$$\frac{i\sin 2n\theta - 2\sin^2 n\theta}{i\sin 2n\theta + 2\cos^2 n\theta} = \frac{-2\sin^2 n\theta + i\,2\sin n\theta\cos n\theta}{2\cos^2 n\theta + i\,2\sin n\theta\cos n\theta}$$

$$\frac{i\sin 2n\theta - 2\sin^2 n\theta}{i\sin 2n\theta + 2(1-\sin^2 n\theta)} = \frac{2\sin n\theta\,[-\sin n\theta - i\cos n\theta]}{2\cos n\theta\,[\cos n\theta + i\sin n\theta]}$$

$$\frac{\cancel{\sin 2n\theta\,(i - 2\sin 2n\theta)}}{\cancel{\sin}} = \tan(n\theta)\, i$$

R.H.S

Example:- Apply de-Moivre's thm to find an equation whose roots are n^{th} powers of roots of equation
$$x^2 - 2x\cos\theta + 1 = 0$$

Sol:- $x^2 - 2x\cos\theta + 1 = 0$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+2\cos\theta \pm \sqrt{(2\cos\theta)^2 - 4}}{2} \Rightarrow \frac{2\cos\theta \pm \sqrt{4\cos^2\theta}}{2}$$

$$= \frac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2}$$

$$= \cos\theta + \sqrt{\cos^2\theta - 1}$$

$$= \cos\theta + \sqrt{-\sin^2\theta} \qquad \sqrt{-1} \Rightarrow i$$

$$\Rightarrow \cos\theta + i\sin\theta$$

$$\text{cis}\,\theta$$

$\beta = \text{cis}(-\theta)$

α^n, β^n

$$(\text{cis}\,\theta)^n \, (\text{cis}(-\theta))^n = (\text{cis}\,n\theta), \, [\text{cis}(-n\theta)]$$

$$x^2 - (\alpha^n + \beta^n)x + \alpha^n\beta^n \Rightarrow 0$$

$$x^2 - [\text{cis}\,n\theta + \text{cis}(-n\theta)]x + \text{cis}\,n\theta \cdot \text{cis}(-n\theta) = 0$$

$$x^2 - [\cos n\theta + i\sin n\theta + \cos(-n\theta) + i\,\cancel{\sin -n\theta}]x + \text{cis}(n\theta - n\theta) \Rightarrow 0$$

$$x^2 - 2x\cos n\theta + \text{cis}\,0 \Rightarrow 0$$

$$x^2 - 2x\cos n\theta + \cos 0 + i\sin 0 \Rightarrow 0$$

$$x^2 - 2x\cos n\theta + 1 \Rightarrow 0$$

Example:- If $x + \frac{1}{x} = 2\cos\theta$, prove that

$$x^n + \frac{1}{x^n} = 2\cos n\theta$$

Sol: $x + \frac{1}{x} = 2\cos\theta$

$$\frac{x^2+1}{x} = 2\cos\theta \Rightarrow x^2+1 = 2x\cos\theta$$

$$x^2 - 2\cos x + 1 = 0$$

$\alpha = \text{cis}\,\theta$

$\beta = \text{cis}(-\theta)$

$L.H.S = x^n + \frac{1}{x^n} =$

$(cis\theta)^n + \frac{1}{(cis\theta)^n}$

$cis\,n\theta + \frac{1}{cis\,n\theta} \Rightarrow cis\,n\theta + cis(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$\cancel{\frac{(cis\,n\theta)^2+1}{cis\,n\theta}}$

$= \cancel{\cos n\theta + i\sin n\theta + cis(n\theta - n\theta)}$

$= \cancel{cis\,n\theta + i\sin n\theta + cis\,0}$

$= \cancel{cis\,n\theta + i\sin n\theta} + 1$

$R.H.S = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$\cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}$

$2\cos n\theta$

$R.H.S = (cis\theta)^n + (cis\theta)^n = 2\cos n\theta$

(4) Prove that for a +ve integer n,

$(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$

hint find polar form of $\sqrt{3}+i$ & $\sqrt{3}-i$

$(\sqrt{3}+i)^n + (\sqrt{3}-i)^n$

$\sqrt{3}+i$

$\sqrt{3} = r\cos\theta$

$1 = r\sin\theta$

$r = \sqrt{(\sqrt{3})^2+(1)^2} = 2$

$\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \tan^{-1}\frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$

$\sqrt{3}+i = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

Now $\sqrt{3}-i \Rightarrow \sqrt{3} = r\cos\theta$

$-1 = r\sin\theta$

$r = \sqrt{(\sqrt{3})^2+(-1)^2} = 2$

$\theta = \tan^{-1}\frac{-1}{\sqrt{3}} = \frac{-\pi}{6}$

$\sqrt{3}-i = 2(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6}))$

$\sqrt{3}-i = 2\{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\}$

$\Rightarrow (\sqrt{3}+i)^n + (\sqrt{3}-i)^n = \cancel{[2(\cos\frac{\pi}{6} + i\sin}$

$\Rightarrow [2(\cos\pi/6 + i\sin\frac{\pi}{6})]^n + [2(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})]^n$

$= 2^n\{\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\} + 2^n[\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}]$

$= 2^n[cis\frac{n\pi}{6} + cis(-\frac{n\pi}{6})]$

$1+i5$

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myCOMPANION

(5) If $(a_1+ib_1)(a_2+ib_2)(a_3+ib_3)\ldots(a_n+ib_n) = A+iB$

(i) prove that

$$(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots(a_n^2+b_n^2) = A^2+B^2$$

Proof $a_1 = r_1\cos\theta_1 \quad a_2 = r_2\cos\theta_2 \quad a_n = r_n\cos\theta_n$

$b_1 = r_1\sin\theta_1 \quad b_2 = r_2\sin\theta_2 \quad b_n = r_n\sin\theta_n$

$r_1^2 = a_1^2+b_1^2 \quad r_2^2 = a_2^2+b_2^2 \quad r_n^2 = a_n^2+b_n^2$

L.H.S: $r_1^2 r_2^2 r_3^2 \ldots r_n^2$

~~R.H.S = A+iB = (r_1 cosθ_1 + r_1 sin~~

R.H.S: $A+iB = (r_1\cos\theta_1 + r_1 i\sin\theta_1)(r_2\cos\theta_2 + r_2 i\sin\theta_2)\ldots(r_n\cos\theta_n + r_n i\sin\theta_n)$

$= r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2) + \ldots r_n(\cos\theta_n + i\sin\theta_n)$

$= r_1 + r_2 \ldots r_n[\operatorname{cis}(\theta_1+\theta_2+\ldots\theta_n)]$

$A+iB = r_1+r_2+\ldots r_n(\cos(\theta_1+\theta_2+\ldots\theta_n)) + r_1+r_2+\ldots r_n(i\sin(\theta_1+\theta_2+\ldots\theta_n))$

$A = r_1+r_2+\ldots r_n(\cos(\theta_1+\theta_2+\ldots\theta_n))$

$B = r_1+r_2+\ldots r_n(i\sin(\theta_1+\theta_2+\ldots\theta_n))$

$A^2+B^2 = r_1^2+r_n^2+\ldots r_n^2$ R.H.S

(ii) $\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \tan^{-1}\frac{b_3}{a_3} + \tan^{-1}\frac{b_4}{4} \ldots \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$

$a_1 = r_1\cos\theta_1$

$b_1 = r_1\sin\theta_1$

$\frac{b_1}{a_1} = \tan\theta_1$

$\tan\frac{b_1}{a_1} = \theta_1+\theta_2+\theta_3$ (sin ... sin ... An

Example:- If $\sin\psi = i\tan\theta$, prove that
$\cos\theta + i\sin\theta = \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$

Proof:- $\sin\psi = i\tan\theta$

~~cos θ = 1~~

$\sin\psi = \frac{i\sin\theta}{\cos\theta}$

$\frac{n}{m} = p$

$\frac{n+m}{n-m} = \frac{p+1}{p-1}$

$\frac{i\sin\theta + \cos\theta}{i\sin\theta - \cos\theta} = \frac{\sin\psi + 1}{\sin\psi - 1}$

$\frac{i\sin\theta + \cos\theta}{i\sin\theta - \cos\theta} = \frac{2\sin\frac{\psi}{2}\cos\frac{\psi}{2} + \cos^2\frac{\psi}{2} + \sin^2\frac{\psi}{2}}{2\sin\frac{\psi}{2}\cos\frac{\psi}{2} - \left(\cos^2\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)}$

$\frac{i\sin\theta + \cos\theta}{i\sin\theta - \cos\theta} = \frac{\left(\sin\frac{\psi}{2} + \cos\frac{\psi}{2}\right)^2}{-\left(\sin\frac{\psi}{2} - \cos\frac{\psi}{2}\right)^2}$

$\frac{\text{cis}\,\theta}{-\text{cis}(-\theta)} =$

$\frac{\text{cis}\,\theta \cdot \text{cis}\,\theta}{-1}$

$-\text{cis}\,\theta = \left(\frac{\tan\frac{\psi}{2} + 1}{-\left(\tan\frac{\psi}{2} - 1\right)}\right)^2$

$(\text{cis}\,\theta)^2 = \left(\frac{\tan\frac{\psi}{2} + 1}{\tan\frac{\psi}{2} - 1}\right)^2$

$$(\text{cis}\theta)^{k} = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

Example: ① $(1+\sin\theta+i\cos\theta)^{n}+(1+\sin\theta+i\cos\theta)^{n} =$

$$2^{n+1}\cos^{n}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)$$

Proof: L.H.S :- $\left[1+\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right]^{n}+\left[1+\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)\right]^{n}$

$= \left[2\cos^{2}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right]^{n}+\left[2\cos^{2}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{2}-\theta\right)\right]^{n}$

$= 2^{n}\cos^{n}\left[\frac{\pi}{4}-\frac{\theta}{2}\right]\left\{\left[\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^{n}+\right.$
$\left.\left[\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^{n}\right\}$

$= 2^{n}\cos^{n}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left\{\text{cis}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)^{n}+\text{cis}\left[-\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^{n}\right\}$

$= 2^{n}\cos^{n}\left[\frac{\pi}{4}-\frac{\theta}{2}\right]\left\{\text{cis}\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)+\text{cis}\left(-\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)\right)\right\}$

$= 2^{n}\cos^{n}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)2\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)$

$= 2^{n+1}\cos^{n}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)$

R.H.S

② $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta+i\cos\theta}\right)^{n} = \cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$

2

Q. How many values does $(\cos\theta + i\sin\theta)^{p/q}$ have? $p, q \in$

Sol: To find all values of $(\cos\theta + i\sin\theta)^{p/q}$ firstly, we find all values of $(\cos\theta + i\sin\theta)^{1/q}$

$\Rightarrow (\cos\theta + i\sin\theta)^{1/q} = [\cos(2n\pi+\theta) + i\sin(2n\pi+\theta)]^{1/q}$ where $n \in Z$

$= \cos\frac{2n\pi+\theta}{q} + i\sin\frac{2n\pi+\theta}{q}, n \in Z_0^+$

$n=0 \quad \cos\theta/q + i\sin\theta/q = \text{cis}\frac{p\theta}{q}$

$n=1 \quad \cos\left(\frac{2\pi+\theta}{q}\right) + i\sin\left(\frac{2\pi+\theta}{q}\right) = \text{cis}\frac{2\pi+\theta}{q}$

$n=2 \quad \cos\left(\frac{4\pi+\theta}{q}\right) + i\sin\left(\frac{4\pi+\theta}{q}\right) = \text{cis}\frac{4\pi+\theta}{q}$

$n=(q-1) \quad \cos\left(\frac{2\pi(q-1)+\theta}{q}\right) + i\sin\left(\frac{2\pi(q-1)+\theta}{q}\right)$

$n=q \quad \cos\frac{2\pi(q)+\theta}{q} + i\sin\frac{2\pi(q)+\theta}{q}$

~~In these question~~

In these q values, we find that angles involved in any two consecutive values differ from one another by $\frac{2\pi}{q} < 2\pi \; (\because q > 1)$. Also we know that no two angle, which differ from one another by less than 2π cannot have their ~~same~~ sine and cosine values same. All the q values obtained are different. Now put $n=q$ & we get $\text{cis}\left(\frac{2\pi(q)+\theta}{q}\right) = \text{cis}\frac{\theta}{q}$

Values at n=0

Illy if we put $n = q+1, q+2 \ldots$ values that we get are same as for $n = 1, 2 \ldots$ so there are only q distinct values of $(\cos\theta + i\sin\theta)^{1/q}$ which can be obt. from $cis\left(\frac{2n\pi+\theta}{q}\right)$ by putting $n = 0, 1, 2 \ldots q-1$

$(\cos\theta + \sin\theta)^{p/q}$, p and q are +ve integer, gcd $(p,q) = 1$

$$(cis\,\theta)^{p/q} = \{(cis\,\theta)^{p}\}^{1/q} = (cis\,p\theta)^{1/q}$$

$$= cis(2n\pi + p\theta), n \in \{0, 1 \ldots\}$$

$$= cis\,\frac{2n\pi + p\theta}{q}, n \in 0, 1, 2 \ldots$$

$n=0$ $\quad cis\,\frac{p\theta}{q}$

$n=1$ $\quad cis\,\frac{2\pi + p\theta}{q}$

$n=2$ $\quad cis\,\frac{4\pi + p\theta}{q}$

$n(q-1)$ $\quad cis\,\frac{2\pi(q-1) + p\theta}{q}$

Q. How many value does $(z)^{45/81}$?

$\frac{p}{q} = \frac{45}{81} = \frac{5}{9} \Rightarrow 9$

n^{th} root of a number

Let $z = a + ib \in \mathbb{C}$ then n^{th} root of z is

$(z)^{1/n}$. It will have n values

Example Find fourth root of unity?

$1 = 1 + 0i$

$1 = r\cos\theta$

$0 = r\sin\theta$

$\left[1^{1/4} = (1\operatorname{cis} 0)^{1/4}\right.$
$0 = r\sin\theta$
$1 + 0 = r^2\cos^2\theta + r^2\sin^2\theta$
$1 = r^2$
$1 = r$

$(1)^{1/4} = (1\operatorname{cis} 0)^{1/4}$

$= (\operatorname{cis}(2n\pi + 0))^{1/4}$; $n \in \{0, 1, 2, 3\}$

$= \frac{2n\pi}{4}$, $n \in 0, 1, 2, 3$

$\operatorname{cis} \frac{n\pi}{2}$, $n = 0, 1, 2 \ldots$

so fourth roots of unity are

$\operatorname{cis} 0$, $\operatorname{cis}\frac{\pi}{2}$, $\operatorname{cis}\pi$, $\operatorname{cis} \frac{3\pi}{2}$

$1, i, -1, -i$

Examples Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their product is 1

Polar form of $\frac{1}{2} + \frac{i\sqrt{3}}{2}$ is $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$$\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{3/4} = \left(\operatorname{cis}\frac{\pi}{3}\right)^{3/4} = \operatorname{cis}\left(3 \times \frac{\pi}{3}\right)^{1/4}$$

$= (\text{cis}\,\pi)^{1/4} = \text{cis}(2n\pi+\pi)^{1/4}$; $n = 0, 1, 2, 3$

$= \text{cis}\,\dfrac{(2n+1)\pi}{4}$; $n = 0, 1, 2, 3$ ...

So the values are $\text{cis}\,\dfrac{\pi}{4}$, $\text{cis}\,\dfrac{3\pi}{4}$, $\text{cis}\,\dfrac{5\pi}{4}$, $\text{cis}\,\dfrac{7\pi}{4}$

$\dfrac{1+i}{\sqrt{2}}$, $\dfrac{-1-i}{\sqrt{2}}$, $\dfrac{-1+i}{\sqrt{2}}$, $\dfrac{1-i}{\sqrt{2}}$

$= \dfrac{(1+i)(-1-i)(-1+i)(1-i)}{4}$

$= \dfrac{(1+1)(1+1)}{4} = 1$

Example: Find n^{th} roots of unity

$(1)^{1/n}$

$1 = r\cos\theta$

$0 = r\sin\theta$

$1 = r^2 \Rightarrow$ $r = 1$

$\tan\theta = 0 \Rightarrow \theta = 0$

$(1)^{1/n} = (\text{cis}\,0)^{1/n}$

$= \text{cis}(2k\pi + 0)^{1/n}$; $k = 0, 1, 2, 3$ ... $(n-1)$

$= \text{cis}\,\dfrac{2k\pi}{n}$, $k = 0, 1, 2, \ldots n-1$

$\text{cis}\,0$, $\text{cis}\,\dfrac{2\pi}{n}$, $\text{cis}\,\dfrac{4\pi}{n}$, $\text{cis}\,\dfrac{6\pi}{n}$ --- $\text{cis}\,\dfrac{2(n-1)\pi}{n}$

$cis\,0,\ cis\frac{2\pi}{n},\ cis\frac{2\pi}{n}\,cis\frac{2\pi}{n},\ cis\frac{2\pi}{n}\,cis\frac{2\pi}{n}\,cis\frac{2\pi}{n}\,cis\frac{2\pi}{n}$

$$cis\frac{2\pi}{n}\left(cis\frac{2\pi}{n}\right)^{n-p}$$

$$\underset{a}{cis\,0},\ \underset{ar}{cis\frac{2\pi}{n}},\ \underbrace{cis\frac{2\pi}{n}\,cis\frac{2\pi}{n}}_{ar^2},\ \underbrace{cis\frac{2\pi}{n}\left(\frac{2\pi}{n}\right)^2}_{ar^3},\ \underbrace{cis\frac{2\pi}{n}\left(\frac{2\pi}{n}\right)}_{ar^4}$$

$$\underbrace{cis\frac{2\pi}{n}\left(cis\frac{2\pi}{n}\right)^{n-b}}_{ar^{n-1}}$$

Ex :- find n^{th} roots of unity and show that they form a G.P also show that sum of all the values is zero, and the product is $(-1)^{n-1}$

Sum of all the values sum of G.P with n terms $=1$, $CR = cis\frac{2\pi}{n}$

$$\frac{a(r^n-1)}{r-1} \text{ or } a\left(\frac{1-r^n}{1-r}\right)$$

$$=(1)\left[\frac{\left(cis\frac{2\pi}{n}\right)^n-1}{cis\frac{2\pi}{n}-1}\right] = \frac{\left(cis\frac{2\pi}{n}\right)^n-1}{cis\frac{2\pi}{n}-1}$$

$$=\frac{cis\frac{2\pi}{n}\times n-1}{cis\frac{2\pi}{n}-1} = \frac{1-1}{cis\frac{2\pi}{n}-1} = 0$$

Q. Product of all the values $=(cis\,0)\left(cis\frac{2\pi}{n}\right)\left(cis\frac{4\pi}{n}\right)$

$$= cis\frac{p(n-1)\pi}{n} = cis\left\{0+\frac{2\pi}{n}+\frac{4\pi}{n}\right.$$

$\text{cis}\,\frac{2\pi}{n}\left[1+2+3+\ldots+(n-1)\right]$

$\text{cis}\,\frac{2\pi}{n}\left[\frac{(n-1)(n-1+1)}{2}\right]$ $\text{cis}\,\frac{2\pi}{n}\left[\frac{(n-1)n}{2}\right]$

$\text{cis}\left[\left(\frac{2\pi}{n}\right)\frac{(n-1)n}{2}\right]$

$\text{cis}\,(n-1)\pi$ Ans

$= \cos(n-1)\pi + i\sin(n-1)\pi$

$= \cos(n-1)\pi + 0$

$= \cos(n-1)\pi$

$= (-1)^{n-1}$

Q. Solution of eq^n's using De-moivres then

solve the eq^n

$x^4 - x^3 + x^2 - x + 1 = 0$

Multiply b/s $(1+x)$

$(1+x)(x^4 - x^3 + x^2 - x + 1) = 0$

$(x^4 - x^3 + x^2 - x + 1) + (x^5 - x^4 + x^3 - x^2 + x) = 0$

$\cancel{x^4} - \cancel{x^3} + \cancel{x^2} - \cancel{x} + 1 + x^5 - \cancel{x^4} + \cancel{x^3} - \cancel{x^2} + \cancel{x} = 0$

$x^5 + 1 = 0$

$x^5 = -1$

$x = (-1)^{1/5}$

OR $x^{10}+x^8+x^6+x^4+x^2+1=0$

multiply b/s $(1-x^2)=0$

$(1-x^2)\, x^{10}+x^8+x^6+x^4+x^2+1=0$

$(x^{10}+x^8+x^6+x^4+x^2+1)-(x^{12}+x^{10}+x^8+x^6+x^4+x^2)$

$\cancel{x^{10}}+\cancel{x^8}+\cancel{x^6}+\cancel{x^4}+\cancel{x^2}+1-x^{12}-\cancel{x^{10}}-\cancel{x^8}-\cancel{x^6}-\cancel{x^4}-\cancel{x^2}=0$

$+1-x^{12}=0$

$1=x^{12}$

$x=(1)^{1/12}$

$=(\text{cis}\,0)^{1/12}$

$=(\text{cis}\,(2n\pi+0))^{1/12}, \quad n=0,1,2,\ldots,11$

$=\text{cis}\,\frac{2n\pi}{12}, \quad n=0,1,\ldots,11$

$=\text{cis}\,\frac{n\pi}{6}, \quad n=0,1,\ldots,11$

Q. Solve the equation $(1+z)^6+z^6=0$

$(1+z)^6+z^6=0$

$(1+z)^6=-z^6$

$\frac{(1+z)^6}{z^6}=-1$

$\left(\frac{1+z}{z}\right)^6=-1$

$$\frac{1+z}{z} = (-1)^{1/6}$$

$$\frac{1}{z} + \frac{z}{z} = (-1)^{1/6}$$

$$\frac{1}{z} + 1 = (-1)^{1/6} \quad \text{—— (1)}$$

$\Rightarrow$ $(-1) = \text{cis}\,\pi$

$$(-1)^{1/6} = (\text{cis}\,\pi)^{1/6}$$

$$= \text{cis}(2n\pi + \pi)^{1/6} \; ; n = 0,1,2,3,4,5$$

$$= \text{cis}\,\frac{2n+1}{6}\pi \; ; \; n = 0,1,2, \ldots 5$$

Put value of $(-1)^{1/6}$ in eq^{n} (1)

$$\frac{1}{z} + 1 = \text{cis}\,\frac{2n+1}{6}\pi$$

$$\frac{1}{z} + 1 = \text{cis}\,\theta \text{ , where } \theta = \frac{2n+1}{6}\pi$$

$$\frac{1}{z} = \text{cis}\,\theta - 1$$

$$z = \frac{1}{\text{cis}\,\theta - 1} \Rightarrow z = \frac{1}{\cos\theta + i\sin\theta - 1}$$

$$z = \frac{1}{(\cos\theta - 1) + i\sin\theta} \Rightarrow \frac{1}{-[1-\cos\theta] + i\sin\theta} \Rightarrow \frac{1}{-2\sin^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$z = \frac{1}{2\sin\theta/2\,(-\sin\frac{\theta}{2} + i\cos\theta/2)}$$

$$z = \frac{1}{2i\sin\theta/2\,(\frac{-1}{i}\sin\theta/2 + \cos\theta/2)}$$

$$z = \frac{1}{2i\sin\theta/2\,(i\sin\theta/2 + \cos\theta/2)}$$

$$z = \frac{1}{2i\sin\theta/_2 \cos\theta/_2}$$

$$= \frac{\cos(-\theta/_2)}{2i\sin\theta/_2} \times \frac{i}{i}$$

$$= \frac{-i\cos(-\theta/_2)}{2\sin\theta/_2}$$

$$= \frac{-i(\cos\theta/_2 - i\sin\theta/_2)}{2\sin\theta/_2}$$

$$= \frac{-i}{2}(\cot\theta/_2 - i)$$

$$= \frac{-i}{2}\cot\theta/_2 + \frac{i^2}{2}$$

$$z = \frac{-1}{2} \frac{-i}{2}\cot\theta/_2$$

where $\theta = \left(\frac{2n+1}{8}\right)\pi$, $n = 0, 1, \ldots 5$

II find the roots of the eq^n

$$(x-1)^7 = x^7$$

$$\frac{(x-1)^7}{(x)^7} = 1$$

$$\left[\frac{x-1}{x}\right]^7 = 1$$

$$\frac{x-1}{x} = (1)^{1/7}$$

$$\frac{x}{x} - \frac{1}{x} = (1)^{1/7}$$

$$1 - \frac{1}{x} = (1)^{1/7} \quad \text{—(1)}$$

$1 = cis\,0$

$(1)^{1/7} = cis(2n\pi + 0)^{1/7}$

$= cis\,\frac{2n\pi}{7}$, where 0, 1, 2 .. 5

Put the value $(1)^{1/7}$ in eqn ①

$1 - \frac{1}{x} = cis\,\theta$ where $\theta = \frac{2n\pi}{7}$

$1 - cis\,\theta = \frac{1}{x}$

$x = \frac{1}{1 - cis\,\theta} \Rightarrow x = \frac{1}{1 - (\cos\theta + i\sin\theta)}$

$x = \frac{1}{(1 - \cos\theta) - i\sin\theta}$

$x = \frac{1}{2\sin^2\theta/2 - i\,2\sin\theta/2\cos\theta/2}$

$= \frac{1}{-2i\sin\theta/2\,(i\sin\theta/2 + \cos\theta/2)}$

$= \frac{1}{-2i\sin\theta/2\,(cis\,\theta/2)}$

$= \frac{cis(-\theta/2)}{-2i\sin\theta/2} \times \frac{i}{i}$

$= \frac{+i\,cis(-\theta/2)}{-2\sin\theta/2} \Rightarrow \frac{-i}{2}\frac{(\cos\theta/2 - i\sin\theta/2)}{\sin\theta/2}$

$= \frac{-i}{2}\{\cos\theta/2 - i)$

$$= \frac{-i}{2}\cot\theta/_2 + \frac{i}{2}$$

$$= \frac{-i}{2} - \frac{i}{2}\cot\frac{\theta}{2}$$

where $\theta = \frac{2n\pi}{7}$

Q $(x-1)^n = x^n$

$$\left(\frac{x-1}{x}\right)^n = 1$$

$$\frac{x-1}{x} = (1)^{1/n}$$

$$\frac{1-1}{x} = (1)^{1/m} (cis\,\theta)^{1/m} = [cis(2r\pi)]^{1/n}$$

II Prove that $\cos^7\theta = \frac{1}{2^6}[\cos 7\theta + 7\cos 5\theta + 21\cos\ldots + 35\cos\theta]$

Let $x = \cos\theta + i\sin\theta = cis\,\theta$ — ①

$\frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$ — ②

Additing ① and ②

$$x + \frac{1}{x} = 2\cos\theta$$

$$\frac{1}{2}\left(x + \frac{1}{x}\right) = \cos\theta$$

$$\left(\frac{1}{2}\right)^7 \left(x + \frac{1}{x}\right)^7 = \cos^7\theta$$

Now, $x^m = (cis\,\theta)^m = cis\ldots$

$= \cos m\theta + i\sin m\theta$

$\frac{1}{x^m} = cis(-m\theta)$

$= \cos m\theta - i\sin m\ldots$

$x^m + \frac{1}{x^m} = 2\cos m\theta$ — ③

$$\left(\frac{1}{2}\right)^7\left(x+\frac{1}{x}\right)^7 = \frac{1}{2^7}\left[{}^7C_0\, x^7 + {}^7C_1\, x^6 \frac{1}{x} + {}^7C_2\, x^5 \frac{1}{x^2} + {}^7C_3\, x^4 \frac{1}{x^3} + {}^7C_4\, x^3 \frac{1}{x^4} + {}^7C_5\, x^2 \frac{1}{x^5} + {}^7C_6\, x \frac{1}{x^6} + {}^7C_7 \frac{1}{x^7}\right]$$

$$= \frac{1}{2^7}\left[x^7 + 7x^5 + 21x^3 + 35x + 35\frac{1}{x} + 21\frac{1}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}\right]$$

$$= \frac{1}{2^7}\left[\left(x^7+\frac{1}{x^7}\right) + 7\left(x^5+\frac{1}{x^5}\right) + 21\left(x^3+\frac{1}{x^3}\right) + 35\left(x+\frac{1}{x}\right)\right]$$

using (1)

$$= \frac{1}{2^7}\left[2\cos 7\theta + (7)(2\cos 5\theta) + (21)(2\cos 5\theta) + (35)2(\cos x\theta)\right]$$

$$= \frac{2}{2^7}\left[\cos 7\theta + 7\cos 5\theta + 21\cos 5\theta + 35\cos x\theta\right]$$

$$= \frac{1}{2^6}\left[\quad " \quad\right] \text{ ans}$$

Prove that $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

Example- Prove that

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$x = cis\theta$

$\frac{1}{x} = cis(-\theta)$

$x - \frac{1}{x} = 2i\sin\theta$

$\left(x-\frac{1}{x}\right)^5 = 2^5 i^5 \sin^5\theta$

$\left(x-\frac{1}{x}\right)^5 = 32 i \sin^5\theta$

Now $\left(x-\frac{1}{x}\right)^5 = {}^5C_0 x^5 + {}^5C_1 x^4\left(-\frac{1}{x}\right) + {}^5C_2 x^3\left(-\frac{1}{x}\right)^2 + {}^5C_3 x^2\left(-\frac{1}{x}\right)^3$

$+ {}^5C_4 x\left(-\frac{1}{x}\right)^4 + {}^5C_5\left(-\frac{1}{x}\right)^5$

$= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$

$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$

$= (2i\sin 5\theta) - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta$

$\left(x-\frac{1}{x}\right)^5 = 2i[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

L.H.S = R.H.S

$32i\sin^5\theta = 2i[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

HH

Example: Prove that

$$\cos^6\theta \sin^4\theta = 2^{-9}[\cos 10\theta + 2\cos 8\theta - 3\cos 6\theta - 8\cos 4\theta + 2\cos 2\theta + 6]$$

$x = cis\,\theta$

$\frac{1}{x} = cis(-\theta)$

$x + \frac{1}{x} = 2\cos\theta$

$x - \frac{1}{x} = 2i\sin\theta$

$$\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^4 = (2\cos\theta)^6(2i\sin\theta)^4$$
$$= 2^6\cos^6\theta\, 2^4\sin^4\theta$$
$$= 2^{10}\cos^6\theta\sin^4\theta$$

$$\text{L.H.S} = \left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^4$$

$$= \left(x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}\right)\left(x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}\right)$$

$$= x^{10} + 6x^8 + 15x^6 + 20x^4 + 15x^2 + 6 + \frac{1}{} - 4x^8 - 24x^6 - 60x^4 - 80x^2 - 60 - \frac{24}{x^2} - \frac{4x^2}{x^4} + 6x^6 + 36x^4 +$$

$$90x^2 + 120 + \frac{90}{x^2} + \frac{36}{x^4} + \frac{6}{x^6} - 4x^4 - 24x^2 -$$

$$60 - \frac{80}{x^2} - \frac{60}{x^4} - \frac{24}{x^6} - \frac{4}{x^8} + x^2 + 6 + \frac{15}{x^2}$$

$$+ \frac{6}{x^8} - \frac{1}{x^{10}}$$

$$= \left(x^{10} + \frac{1}{x^{10}}\right) + \left(2x^8 + \frac{2}{x^8}\right) + \left(-3x^6 - \frac{3}{x^6}\right) +$$

$$+\left(8x^4 - \frac{8}{x^4}\right) + \left(2x^2 + \frac{2}{x^2}\right) + 12$$

$$= 2\cos 10\theta + 4\cos 8\theta - 6\cos 6\theta - 16\cos 4\theta + 4\cos 2\theta + 12$$

Ques iii Express $\cos^4\theta \sin^3\theta$ in a series of sines of multiples of θ.

Sol!

$$\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3 = (2\cos\theta)^4(2i\sin\theta)^3$$

$$= -2^7 i \cos^4\theta \sin^3\theta$$

$$\cos^4\theta \sin^3\theta = \frac{-1}{2^7 i}\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3$$

$$= \frac{-1}{2^7 i}\left[{}^4C_0 x^4 + {}^4C_1 x^3 \frac{1}{x} + {}^4C_2 x^2\left(\frac{1}{x}\right)^2 + {}^4C_3 x\left(\frac{1}{x}\right)^3 + {}^4C_4\left(\frac{1}{x}\right)^4\right]\left[{}^3C_0 x^3 + {}^3C_1 x^2\left(\frac{-1}{x}\right) + {}^3C_2\, x\left(\frac{-1}{x}\right)^2 + {}^3C_3\left(\frac{-1}{x}\right)^3\right]$$

$$= \frac{-1}{2^7 i}\left[x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}\right]\left[x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}\right]$$

$$= \frac{-1}{2^7 i}\left[x^7 - 4x^5 - 6x^5 - 4x + 4x^5 - 16x^3 - 24x - \frac{24}{x} + 6x^3 - 24x - \frac{36}{x} - \frac{24}{x^3} + 4x - \frac{16}{x} - \frac{24}{x^3} - \frac{16}{x^5} + \frac{4}{x} - \frac{4}{x^3} - \frac{3}{x^5} - \frac{1}{x}\right]$$

$= x^7 - 3x^5 + 3x^3 - x + 4x^5 - 12x^3 + 12x - \frac{4}{x} + 6x^3 - 18x + \frac{18}{x} - \frac{6}{x^3}$
$+ 4x - \frac{12}{x} + \frac{12}{x^3} - \frac{4}{x^5} + \frac{1}{x} - \frac{3}{x^3} + \frac{3}{x^5} - \frac{1}{x^7}$

$= \cancel{\frac{1}{2^7 i}} \left[\cancel{x^7} - \cancel{16x^3} + \left(-3x^3 + \frac{3}{x^3}\right) + \left(-3x + \frac{3}{x}\right) \right]$

$= \left(x^7 - \frac{1}{x^7}\right) + \left(x^5 - \frac{1}{x^5}\right) - 3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$

Example:- Express $\cos 7\theta$ and $\sin 7\theta$ in powers of $\sin\theta$ and $\cos\theta$. Also find $\tan 7\theta$ in terms of powers of $\tan\theta$.

Sol:- $(\cos\theta + i\sin\theta)^7 = \operatorname{cis} 7\theta$
$\cos 7\theta + i\sin 7\theta$

$(\cos\theta + i\sin\theta)^7 = {}^7C_0 \cos^7\theta + {}^7C_1 \cos^6\theta\, i\sin\theta + {}^7C_2 \cos^5\theta (i\sin\theta)^2 + {}^7C_3 \cos^4\theta (i\sin\theta)^3 + {}^7C_4 (i\sin\theta)^4 + {}^7C_5 \cos^2\theta (i\sin\theta)^5 + {}^7C_6 \cos\theta (i\sin\theta)^6 + {}^7C_7 (i\sin\theta)^7$

$\Rightarrow \cos 7\theta + i\sin\theta = \cos^7\theta + 7i\cos^6\theta \sin\theta - 21\cos^5\theta \sin^2\theta - 35i\cos^4\theta \sin^3\theta + 35\cos^3\theta \sin^4\theta + 21i\cos^2\theta \sin^5\theta - 7\cos\theta \sin^6\theta - i\sin^7\theta$

$\Rightarrow \cos 7\theta = \cos^7\theta - 21\cos^5\theta \sin^2\theta + 35\cos^3\theta \sin^4\theta - 7\cos\theta \sin^6\theta$. — (1)

$\sin 7\theta = 7\cos^6\theta \sin\theta - 35\cos^4\theta \sin^3\theta + 21\cos^2\theta \sin^5\theta - \sin^7\theta$. — (2)

Divide (2) by (1)

$$\tan 7\theta \; \frac{\sin 7\theta}{\cos 7\theta} = \frac{7\cos^6\theta \sin\theta - 35\cos^4\theta \sin^3\theta + 21\cos^2\theta \sin^5\theta - \sin^7\theta}{\cos^7\theta - 21\cos^5\theta \sin^2\theta + 35\cos^3\theta \sin^4\theta - 7\cos\theta \sin^6\theta}$$

$$\tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

Example Prove that $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{8\pi}{7}$ are roots of eq^n $8x^3+4x^2-4x-1=0$

Proof:- Let $\theta = \frac{2n\pi}{7}$ $n \in$ — ①

$n=3$, $\theta = \frac{6\pi}{7}$ $\quad \cos\frac{6\pi}{7} = \cos\left(\frac{6\pi}{7}-2\pi\right)$

$= \cos\frac{-8\pi}{7} = \cos\frac{8\pi}{7}$

$n=5$: $\theta = \frac{10\pi}{7}$ $\quad \cos\frac{10\pi}{7} = \cos\left[\frac{10\pi}{7}-2\pi\right)$

$= \cos\left(\frac{-4\pi}{7}\right) = \cos\frac{4\pi}{7}$

So new values of $\cos\theta$ are obtained only for $n=1,2,7$

using ① $7\theta = 2n\pi$

$4\theta+3\theta = 2n\pi$

~~$4\theta+3\theta =$~~

$4\theta = 2n\pi-3\theta$

$\cos 4\theta = \cos(2n\pi-3\theta)$

$= \cos(-3\theta)$

$= \cos(3\theta)$

$\cos 4\theta = \cos 3\theta$

$2\cos^2 2\theta - 1 = -3\cos\theta + 4\cos^3\theta$

$2(\cos 2\theta)^2 - 1 = -3\cos\theta + 4\cos^3\theta$

$2(2\cos^2\theta-1)^2 - 1 = -3\cos\theta + 4\cos^3\theta$

$2[4\cos^4\theta - 4\cos^2\theta + 1] - 1 = -3\cos\theta + 4\cos^3\theta$

$8\cos^4\theta - 8\cos^2\theta + 2 - 1 = -3\cos\theta + 4\cos^3\theta$

$8\cos^4\theta - 4\cos^3\theta - 8\cos^2\theta + 3\cos\theta + 1 = 0$

$8x^4 - 4x^3 - 8x^2 - 3x + 1 = 0$

[Put $\cos\theta = x$ $\quad$ ~~$x = \cos\theta$~~

substitute $x = -1$

$(x+1)(8x^3+4x^2-4x+1)=0$

$8x^3+4x^2-4x+1=0$ — (1)

and $\cos\theta$ satisfies eqn (1)

$\cos 2n\pi/7$ is roots of eqn (1)

$\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{8\pi}{7}$

Ans

$$
\begin{array}{r}
8x^3+4x^2-4x+1 \\
x+1 \overline{)\, 8x^4+4x^3-8x^2-3x+1} \\
8x^4+8x^3 \\
\hline
-4x^3-8x^2-3x+1 \\
-4x^3-4x^2 \\
\hline
-4x^2-3x+1 \\
-4x^2-4x \\
\hline
x+1 \\
x+1 \\
\hline
\times
\end{array}
$$

example Let the eqn ax^2+bx+c have roots α & β find the eqn with roots $\frac{1}{\alpha}, \frac{1}{\beta}$.

Sol:- construct a new eqn whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

Suppose new eqn is in variable y

$y=\frac{1}{\alpha}$, $y=\frac{1}{\beta}$

$x=\alpha$
$x=\beta$

$y=\frac{1}{x}$ $\qquad y=\frac{1}{x}$

so

$x=\frac{1}{y}$, $x=\frac{1}{y}$

$a\left(\frac{1}{y}\right)^2+b\left(\frac{1}{y}\right)+c=0$

$a+by+cy^2=0$

$cy^2+by+a=0$

example

ex

example Construct an eq^n whose roots are

$$\sec^2\frac{2\pi}{7},\ \sec^2\frac{4\pi}{7},\ \sec^2\frac{8\pi}{7}$$

Ans $y^3-24y^2+80y-64=0$

Sol- eq^n with roots $\sec\frac{2\pi}{7}, \sec\frac{4\pi}{7}, \sec\frac{8\pi}{7}$ is

$$y^3-4y^2-4y-8=0$$

Let $\theta = \frac{2\pi}{7}$

(2)

example:- $8x^3+4x^2-4x-1=0$. If its roots are $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$ & $\cos\frac{8\pi}{7}$. find the equation, whose roots are

(a) $\sec\frac{2\pi}{7}$, $\sec\frac{4\pi}{7}$ & $\sec\frac{8\pi}{7}$

Sol: Let the eq^n be in variable y
roots of eq^n (1) are $\alpha = \cos\frac{2\pi}{7}$, $\beta = \cos\frac{4\pi}{7}$
$\gamma = \cos\frac{8\pi}{7}$

Then roots of the new eq^n are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$
$x = \alpha, \beta, \gamma$

$y = \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

In each case
$y = \frac{1}{x} \rightarrow x = \frac{1}{y}$

Put value of x in (1)

$$\frac{8}{y^3} + \frac{4}{y^2} - \frac{4}{y} - 1 = 0$$

$$8+4y-4y^2-y^3=0$$

$\boxed{y^3+4y^2-4y-8=0}$ which is req eq^n.

$8x^3+4x^2-4x-1=0 \rightarrow \cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{8\pi}{7}$

(b) for $\sec^2\frac{2\pi}{7}$, $\sec^2\frac{4\pi}{7}$, $\sec^2\frac{8\pi}{7}$

$$y = \frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$$

In each case $y = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{y}$ $\therefore x = \frac{1}{\sqrt{y}}$

Put the value of x in (1)

$$\frac{8}{(\sqrt{y})^3} + \frac{4}{(\sqrt{y})^2} - \frac{4}{\sqrt{y}} - 1 = 0$$

$$\frac{8+4\sqrt{y}-4y-y\sqrt{y}}{y\sqrt{y}}=0$$

$8+4\sqrt{y}-4y-y\sqrt{4}=0$

$(8-4y)+\sqrt{y}(4-y)=0$

$(8-4y)=-\sqrt{y}(4-y) \Rightarrow (8-4y)^2=y(4-y)^2$

$(64+16y^2-64y)=(y\,16+y^2-8y)$

$64+16y^2-64y=y(16+y^2-32y)$

$64+16y^2-64y=16y+y^3-32y^2$

$16y-8y^2+y^3-64+64y-16y^2=0$

$y^3-24y^2+80y-64=0$

Exercise: Given the $\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$ and $\sec^2\frac{5\pi}{7}$ are roots of eq^n

$\boxed{1+\tan^2\theta=\sec^2\theta}$

$x^3-24x^2+80x-64=0$

find can eq^n whose roots are

$\tan^2\frac{\pi}{7}$, $\tan^2\frac{3\pi}{7}$ and $\tan^2\frac{5\pi}{7}$

Sol: $\alpha=1+\tan^2\frac{\pi}{7}$ $\qquad$ $\beta-1=\tan^2\pi/7$

$\alpha-1=\tan^2\pi/7$

$\gamma-1=\tan^2\pi/7$

$x-1=y$

$x=1+y$

$x=(1+y)^3-24(1+y)^2+80(1+y)-64=0$

$=1+y^3+3y^2+3y-24(1+y^2+2y)+80+80y-64$

$=1+y^3+3y^2+3y-24-24y^2-48y+80+80y-64$

$=y^3-21y^2+35y-7=0$

Ans

Q. Also deduce that $\cot^2\frac{\pi}{7}, \cot^2\frac{3\pi}{7}, \cot^2\frac{5\pi}{7} = \frac{1}{7}$

$$\left(\tan^2\frac{\pi}{7}\right)\left(\tan^2\frac{3\pi}{7}\right)\left(\tan^2\frac{5\pi}{7}\right) = \frac{-(-7)}{1} = 7$$

$$\therefore \boxed{(\alpha)(\beta)(\gamma) = \frac{-d}{a}}$$

$$\therefore \left(\cot^2\frac{\pi}{7}\right)\left(\cot^2\frac{3\pi}{7}\right)\left(\cot^2\frac{5\pi}{7}\right) = \frac{1}{7}$$ Ans

Unit-2

Rank of a Matrix :- A no. r is said to be rank of a non-zero matrix A if

1. ∃ atleast one minor of order r of A (Nonzero) which does not vanish.
2. Every minor of order (r+1) (if any), vanishes

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 6 & 7 & 3 & 9 \end{bmatrix}$$

Q1) $$A = \begin{bmatrix} -8 & 7 & 2 & 3 \\ -2 & 3 & 0 & 1 \\ -3 & 2 & 1 & 1 \\ 0 & 6 & 6 & 1 \end{bmatrix}$$

$$|A| = -8\begin{vmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ 6 & 6 & 1 \end{vmatrix} - 7\begin{vmatrix} -2 & 0 & 1 \\ -3 & 1 & 1 \\ 0 & 6 & 1 \end{vmatrix} +$$

$$2\begin{vmatrix} -2 & 3 & 1 \\ -3 & 2 & 1 \\ 0 & 6 & 1 \end{vmatrix} - 3\begin{vmatrix} -2 & 3 & 0 \\ -3 & 2 & 1 \\ 0 & 6 & 6 \end{vmatrix}$$

$$\Rightarrow -8\left[3(1-6)-0(2-6)+1(12-6)\right]-7\left[-2(1-6)-0[-3\ldots +1[-18-0)\right]+2\left[-2(2-6)-3[-3-0)+1[-18-0]\right]$$

$$= -8\left[3-18+12-6\right]-7\left[-2+\cancel{12}+\cancel{18}\right]+2\left[-4+12+9-18\right]$$

$$= -8\left[-24+15\right]-7\left[-2\right]+2\left[-22+21\right]+$$

Rank (4)

$= +8[$

(2) $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 5 & 2 \\ 12 & -4 & 5 \end{bmatrix}$

Sol: $= 4\begin{bmatrix} 5 & 2 \\ -4 & 5 \end{bmatrix} - 2\begin{bmatrix} 8 & 2 \\ 12 & 5 \end{bmatrix} + 3\begin{bmatrix} 8 & 5 \\ 12 & -4 \end{bmatrix}$

$= 4[25+8] - 2[40-24] + 3[-32-60]$

$= 4(34) - 2(16) + 3[-92]$

$= 136 - 32 - 276$

$= 136 - 308$

$= -172 \neq 0$

Rank $= 3$

(3) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 4 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \Rightarrow (-1-0) = -1 \neq 0$ Rank $= 2$

(4) $\begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -4 & 1 & 2 \\ 5 & 2 & 1 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & -3 \\ 3 & -4 & 1 \\ 5 & 2 & 1 \end{bmatrix} \Rightarrow (-4-2) - 2(3-5) - 3(6+20)$

$-6 + 4 - 76$

$= 4 - 82$

$= -78 \neq 0$ Rank $= 3$

92
3
276

316
40
24
16

60
32
92

34
4
136

276
32
308

136
172

myCOMPANION

(5) $A = \begin{bmatrix} a & 0 & 1 \\ 1 & 2 & a \\ 1 & 2 & 3 \end{bmatrix}$

$= a\begin{bmatrix} 2 & a \\ 2 & 3 \end{bmatrix} - 0\begin{bmatrix} 1 & a \\ 1 & 3 \end{bmatrix} + 1\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$= a[6-2a] - 0[3-a] + 1[2-2]$

$= a[6-2a] + 1(0)$

$= \cancel{6a-2a^2}$

$(a)(6-2a) = 0$

$a = 0, 3$

(6) Find the value of x so that matrix

$A = \begin{bmatrix} x+p & q & r \\ p & x+q & r \\ p & q & x+r \end{bmatrix}$ is of rank 3

$= x+p+q+r \begin{bmatrix} 1 & q & r \\ 1 & x+q & r \\ 1 & q & x+r \end{bmatrix}$

$= x+p+q+r \begin{bmatrix} 1 & q & r \\ 0 & x & 0 \\ 0 & -x & x \end{bmatrix}$

$(x+p+q+r)(1)\, x^2 \neq 0$

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(7) Find the value of x so that rank of matrix

$$A = \begin{bmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{bmatrix} \text{ is } \leq 2$$

Sol:- $\cancel{3x-8+6} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3x-8 & 3 \\ 1 & 3 & 3x-8 \end{bmatrix}$

$x-8+3+3$

$\cancel{3x-8+6}\ (1) \begin{bmatrix} 3x-8 & 3 \\ 3 & 3x-8 \end{bmatrix}$

$\cancel{3x-8+6}\ (3x-8)^2 - 9$

$9x^2+64-48x-9=0$

$9x^2-48x+55=0$

$x = \frac{2}{3}, \frac{5}{3}\quad \rho(A)=2$

$x = \frac{11}{3},\ \rho(A)=1$

(8) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ find rank of matrix A using elementary transformations.

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{bmatrix}$ $R_2 = R_2 - R_1,\ R_3 \Rightarrow R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{bmatrix}$ $C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad R_3 \to R_3 - \frac{1}{2} R_2$$

(9) $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 3 & 5 & 4 \end{bmatrix}$ find roots of matrix using elementary transforma

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -6 & -3 \\ 0 & -4 & -2 \end{bmatrix} \quad R_2 \to R_2 - 4R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & -3 \\ 0 & -4 & -2 \end{bmatrix} \quad C_2 \to C_2 - 3C_1, C_3 \to C_3 - 2C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & -4 & 0 \end{bmatrix} \quad C_3 \to C_3 - \frac{1}{2} C_2$$

$C_3 \to C_3 - \frac{1}{2} C_2$

$$\rho(A) = 2$$

(10) $A = \begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \\ 3 & -1 & -1 & 7 \end{bmatrix}$ find roots of matrix A using elementary transformation

$$\sim \begin{bmatrix} 1 & 4 & 1 & 5 \\ 0 & -3 & -3 & -12 \\ 0 & 2 & -4 & -8 \end{bmatrix} \quad R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -3 & -12 \\ 0 & 2 & -4 & -8 \end{bmatrix} \quad C_2 \to C_2 + C_1, C_3 \to C_3 - C_1$$

$C_4 \to C_4 - 5C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -3 & -12 \\ 0 & 0 & -2 & 0 \end{bmatrix} \quad R_3 \to R_3 - \frac{2}{3} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \quad C_3 \to C_3 + C_2,\ C_4 \to C_4 + 4C_2$$

Rank = 2

(11) $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -1 \end{bmatrix}$ using elementary transformations find the rank of matrix.

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & 1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -1 \end{bmatrix}$$

$R_2 \to 2R_1 - R_2$, $R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & +5 & 3 & 9 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 7 & -11 \\ 0 & 4 & -1 & 6 \\ 0 & 24 & 43 & -98 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 + C_1, \\ C_3 \to 2C_2 + C_1 \\ C_4 \to C_4 - 4C_2 \end{matrix}$$

-8 -2
-2
+3 -6 -3
3 - 27
-1+1
-2
-46+3
-4
8-9
-2-(-2)
-2+2
10-3

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 9 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \quad \begin{aligned} &C_2 \to C_2 + C_1, \\ &C_3 \to C_3 + 2C_1, \\ &C_4 \to C_4 + 4C_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 9 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_3 \to R_3 - \frac{4}{5}R_1$, $R_4 \to R_4 - \frac{9}{5}R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 9 \\ 0 & 0 & 33/5 & 14/5 \\ 0 & 0 & 33/5 & 4/5 \end{bmatrix}$$

$C_3 \to C_3 - \frac{3}{5}C_2$, $C_4 \to C_4 - \frac{9}{5}C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 33/5 & \\ 0 & & & \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 33/5 & 14/5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 33/5 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad C_4 \to C_4 - \frac{14}{33}C_3$$

$4 - \frac{1}{5}$

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(12) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & x \\ 2 & 2x-2 & -x-2 & 3x-1 \\ 3 & x+2 & -3 & 2x+1 \end{bmatrix}$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - 3R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & x-1 \\ 0 & 2x-4 & x-4 & 3x-3 \\ 0 & x-1 & -6 & 2x-2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & x-1 \\ 0 & 2x-4 & -x-4 & 3x-3 \\ 0 & x-1 & -6 & 2x-2 \end{bmatrix}$ $C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1,\ C_4 \to C_4 - C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & x-1 \\ 0 & 0 & 2x-10 & (x-1)(5-x) \\ 0 & 0 & 3x-12 & \frac{(x-1)(5-x)}{2} \end{bmatrix}$

$R_3 \to R_3 - (x-2)R_2,\ R_4 \to R_4 - \left(\frac{x-1}{2}\right)R_2$

$\cancel{\begin{matrix} -x-4+(x-2)(3) \\ -x-4+3x-6 \\ 2x-10 \end{matrix}}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2(x-5) & (x-1)(5-x) \\ 0 & 0 & \frac{3}{2}(x-5) & \frac{(x-1)(5-x)}{2} \end{bmatrix}$

$C_3 \to C_3 + \frac{3}{2}C_2,\ C_4 \to C_4 - \frac{(x-1)}{2}C_2$

$R_4 \to R_4 - \frac{3}{4} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2(x-5) & (x-1)(5-x) \\ 0 & 0 & 0 & \frac{-(x-1)(5-x)}{4} \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2(x-5) & 0 \\ 0 & 0 & 0 & \frac{-(x-1)(5-x)}{4} \end{bmatrix} \quad C_4 \to C_4 + \frac{1}{2}(x-1)$$

$\rho(A) = 4$ if $x \neq 1, x \neq 5$

$\rho(A) = 3$ if $x = 1$

$\rho(A) = 2$, if $x = 5$

Normal form of a matrix

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_r & 0 \end{bmatrix}, \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} I_r \end{bmatrix}$$

eg. $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \sim \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$.

Q. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & 0 \end{bmatrix}$$

Q Reduce matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ to normal form

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

~~$C_3 \to C_2 + 3C_3$~~

$C_3 \to C_3 + 2C_1$

$C_1 \to C_1 + 2C_2$

$0 + 2(1)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$2 - 2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 -$

$C_3 \to C_3 - C_2$

$2 - 1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim [I_3] \quad \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_{2\times 2} & 0 \\ 0 & 0 \end{bmatrix}$$ Ans

Reduce $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ to normal form

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 0 \\ 4 & -2 & -1 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 + C_2 \\ C_4 \to C_4 + C_2 \end{matrix}$$

$$\begin{bmatrix} \cancel{0} & \cancel{-2} & \cancel{-1} & \cancel{4} \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_3 \\ \cancel{R_1 \to R_1} \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 4 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad C_2 \to C_2 - C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$C_1 \to C_1 - C_4$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \to C_3 - C_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim [I_2]$$

~~Th Rotate A~~

Echelon ~~of~~ Form of a matrix

A matrix A is said to be in row echelon from if

(I) The zero rows of A occur below all the non-zero rows of A.

(II) The no. of zero before the first non-zero entry in a row is less than or equal to no. of such zeroes in next row.

III If $R_1, R_2, \ldots$ are non-zero rows of A then the first non-zero entry in these row is 1.

Reduce the matrix to echelon form

$$\begin{bmatrix} 1 & -2 & -1 & 4 \\ 2 & -4 & 3 & 5 \\ -1 & 2 & 6 & 7 \end{bmatrix}_{3\times 4}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & 11 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_2 \to \frac{1}{5}R_2, \quad R_3 \to \frac{1}{14}R_3$$

with column transformation

$$\sim \begin{bmatrix} 1 & -2 & -1 & 4 \\ 2 & -4 & 3 & 5 \\ -1 & 2 & 6 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 2 & 0 & 5 & 5 \\ -1 & 0 & 5 & 7 \end{bmatrix} \quad C_2 \to C_2 + 2C_1, \quad C_3 \to C_3 + C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 2 & 0 & 1 & 5 \\ -1 & 0 & 1 & 7 \end{bmatrix} \quad \begin{matrix} C_3 \to 2C_3 - C_2 \\ C_4 \to C_4 - 4C_1 \\ 7 - 4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ -1 & 0 & 4 & 3 \end{bmatrix} \quad \begin{matrix} C_4 \to 2C_4 - 2C_1 \\ 8 + 1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 0 & 4 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} C_3 \to \frac{1}{4} C_3 \\ C_4 \to \frac{1}{7} C_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \quad C_4 \to C_4 - C_3$$

$$C_2 \leftrightarrow C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

\# $$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 1 & 4 & 2 \\ 1 & -3 & 6 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -5 & 7 & 2 \\ 1 & -5 & 7 & 2 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1 \end{matrix}$$

$(a^2+b^2+c^2)^2 = 2ab + 2bc + 2ca + a^2 + b^2 + c^2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - C_3 \\ C_3 \to C_3 - C_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \neq 1 \qquad \rho = 2$$

Prove that

$$\frac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2$$

where $x = 2\cos\theta$

$(x^3 - x^2 - 2x + 1)^2 (1+\cos\theta)$

$= (8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1)^2 (1+\cos\theta)$

$= (8\cos^3\theta)^2 + (-4\cos^2\theta)^2 + (-4\cos\theta+1)^2 + 2(8\cos^3\theta)(-4\cos^2\theta) + 2(-4\cos\theta+1)(-4\cos^2\theta) + 2(-4\cos\theta+1)(8\cos^3\theta)$
$(1+\cos\theta)$

$= 64\cos^6\theta + 16\cos^4\theta + 16\cos^2\theta + 1\overset{-8\cos\theta}{\cancel{6}} - 64\cos^5\theta \cancel{+32\cos^3}$
$+ 8\cos^2\theta(-4\cos\theta+1) + 16\cos^3\theta(-4\cos\theta+1)(1+\cos\theta)$

$= 64\cancel{\cos^6\theta} + 16\cos^4\theta + 16\cos^2\theta + 1\overset{-8\cos\theta}{} - 64\cancel{\cos^5\theta} + 32\cos^3\theta -$

$16-64+32$

$$8\cos^2\theta - 64\cos^4\theta + 16\cos^3\theta \,](1+\cos\theta)$$

$$= 16\cos^4\theta + 16\cos^2\theta + 1 - 8\cos\theta + 32\cos^3\theta - 8\cos^2\theta - 64\cos^4\theta + 16\cos^3\theta + 16\cos^4\theta$$

$$= \cancel{16\cos^4\theta + 16\cos^2\theta} -$$

$$= 64\cos^6\theta + 16\cos^4\theta + 16\cos^2\theta + 1 - 8\cos\theta - 64\cos^5\theta + 32\cos^3\theta - 8\cos^2\theta - 64\cos^4\theta + 16\cos^3\theta + 64\cos^7\theta + 16\cos^5\theta + 16\cos^3\theta + \cos\theta - 8\cos^2\theta - 64\cos^6\theta + 32\cos^4\theta - 8\cos^3\theta - 64\cos^5\theta + 16\cos^4\theta$$

$$= 16\cos^4\theta - 64\cos^4\theta + 32\cos^4\theta - 64\cos^5\theta + 16\cos^4\theta - 64\cos^5\theta + 16\cos$$

$$= 32\cos^4\theta - 64\cos^4\theta + 32\cos^4\theta - 64\cos^5\theta + 16\cos^5\theta - 64\cos^5\theta + 64\cos^7\theta + 32\cos^3\theta + 16\cos^3\theta + 16\cos^3\theta - 8\cos^3\theta + 16\cos^2\theta - 8\cos^2\theta - 8\cos^2\theta$$

$$64\cos^4\theta - 64\cos^4\theta - 128\cos^5\theta + 16\cos^5\theta + 64\cos^7\theta + 58\cos^3\theta + 16\cos^2\theta - 16\cos^2\theta$$

$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta + 1 - 8\cos\theta + \cos\theta$$
$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta + 1$$

$$(1+\cos 7\theta) \Rightarrow 1 + [\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta]$$

Vectors

Vector:- A vector is an ordered set of number and if it is set of n number then it is called n-vector.

Linearly depended and Linearly Independent vectors:-

vectors $v_1, v_2 \ldots v_n$ are called linearly depended, if there exist scalars $x_1, x_2 \ldots x_n$ not all zero such that

$$x_1 v_1 + x_2 v_2 + \ldots + x_n v_n = 0$$

Vectors $v_1, v_2 \ldots v_n$ are called linearly independed if whenever $x_1 v_1 + x_2 v_2 + \ldots + x_n v_n$ then $x_1 = x_2 \ldots x_n = 0$

If $\exists$ atleast one non-zero α_i such that $\alpha_1 v_1 + x_2 v_2 + \ldots + \alpha_n v_n = 0$ then $v_1 v_2 \ldots v_n$ are linearly depended

\# Show that the vectors

(i) $[2\ 3\ {-1}\ {-1}], [1\ {-1}\ {-2}\ {-4}], [3\ 1\ 3\ {-2}], [6\ 3\ 0\ {-7}]$ are LD

(ii) Also express any one vector as linear combination of other vector

Sol/n Consider

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0 \qquad \alpha_i\text{'s scalar}$$

$$\alpha_1[2\ 3\ {-1}\ {-1}] + \alpha_2[1\ {-1}\ {-2}\ {-4}] + \alpha_3[3\ 1\ 3\ {-2}] + \alpha_4[6\ 3\ 0 \ldots$$

$$(2\alpha_1, 3\alpha_1, -1\alpha_1, -1\alpha_1) + (\alpha_2, -\alpha_2, -2\alpha_2, -4\alpha_2) + (3\alpha_3, \alpha_3, 3\alpha_3, -2\alpha_3) \ldots$$

$+ [6\alpha_4 \; 3\alpha_4 \; 0 \; -7\alpha_4] = 0\,0\,0\,0$

$[2\alpha_1+\alpha_2+3\alpha_3+6\alpha_4 \quad 3\alpha_1-\alpha_2+\alpha_3+3\alpha_4 \quad -1\alpha_1-2\alpha_2+3\alpha+0$
$-1\alpha_1-4\alpha_2-2\alpha_3-7\alpha_4] = [0 \; 0 \; 0 \; 0]$

$2\alpha_1+\alpha_2+3\alpha_3+6\alpha_4 = 0$ — (1)
$3\alpha_1-\alpha_2+\alpha_3+3\alpha_4 = 0$ — (2)
$-\alpha_1-2\alpha_2+3\alpha+0 = 0$ — (3)
$-\alpha_1-4\alpha_2-2\alpha_3-7\alpha_4 = 0$ — (4)

Add (1) and (2) eq

$5\alpha_1+4\alpha_3+9\alpha_4 = 0$ — (5)

Sub $2(2)-(3)$ eq

$7\alpha_1-\alpha_3+6\alpha_4 = 0$ — (6)

Subtract $2(3)-(4)$ eq

$-\alpha_1+8\alpha_3+7\alpha_4 = 0$ — (7)

Sub $5+4(6)$

$5+4$

$33\alpha_1+33\alpha_4 = 0$ — 8

Sub $8(6)-(7)$

$55\alpha_1+55\alpha_4 = 0$ — (9)

$55[\alpha_1+\alpha_4] = 0$

$\alpha_1+\alpha_4 = 0$

$\alpha_4 = -\alpha_1$ — Put in (5)

from (5) $5\alpha_1+4\alpha_3-9\alpha_1 = 0$

$4\alpha_3-4\alpha_1 = 0 \qquad \alpha_3 - \alpha_1 \Rightarrow \qquad \alpha_3 = +\alpha_1$ — Put (1)

$= I \cdot \phi 1$
fl

$\alpha_4 = -\alpha_1$, $\alpha_3 = \alpha_1$ Put in (1)

$2\alpha_1 + \alpha_2 + 3\alpha_1 + 6\alpha_1 = 0$

$\alpha_2 - 1\alpha_1 = 0$

$\alpha_2 = \alpha_1 = \alpha_3$

$-\alpha_1 = \alpha_4$

from (1)

$2\alpha_1 + \alpha_2 + 3\alpha_3 + 6\alpha_4 = 0$

$2\alpha_1 + \alpha_1 + 3\alpha_1 - 6\alpha_1 = 0$

$\alpha_1 = 0$

(ii) $\alpha_1 = k \neq 0$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$

$\alpha_1 v_1 + \alpha_1 v_2 + \alpha_1 v_3 - \alpha_1 v_4 = 0$

$\alpha_1 [v_1 + v_2 + v_3 - v_4] = 0$

$v_1 + v_2 + v_3 - v_4 = 0$

$v_1 + v_2 + v_3 = v_4$

Other method :-

$[2\ 3\ -1\ -1]$ $[1\ -1\ -2\ -4]$, $[3\ 1\ 3\ -2]$,
$[6\ 3\ 0\ -7]$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$

$[2\alpha_1 + \alpha_2 + 3\alpha_3 + 6\alpha_4 \quad 3\alpha_1 - \alpha_2 + \alpha_3 + 3\alpha_4 \quad -\alpha_1 - 2\alpha_2$
$- \alpha_1 - 4\alpha_2 - 2\alpha_3 - 7\alpha_4] = 0$

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = 0$$

$AX = 0$

$X = 0$

case I :- $|A| \neq 0$

A^{-1} exists

$A \cdot X = 0$

Implies $A^{-1}AX = 0$

$X = 0$

So vector are L.I

case II :- $|A| = 0$

So A^{-1} does not exist

then vector are L.D

Check whether the vector $(2\ {-1}\ 3)$, $(8\ 2\ 0)$ and $(0\ 1\ {-2})$ are L.I or L.D

$[2\ {-1}\ 3]$, $[8\ 2\ 0]$, $[0\ 1\ {-2}]$

$$|A| = \begin{bmatrix} 2 & -1 & 3 \\ 8 & 2 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$|A| = 2(-4) + 1(-16) + 3(8)$

$= -8 - 16 + 24 = 0$

Hence vector are depended.

11

write the vector (3 2 1) as a linear combination of vectors $v_1 = (2\ -1\ 0)$, $v_2 = (1\ 2\ -1)$ and $v_3\ (0\ 2\ -1)$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = [3\ 2\ 1)$$

$$\alpha_1[2\ -1\ 0] + \alpha_2[1\ 2\ -1) + \alpha_3[0\ 2\ -1] = 3\ 2\ 1$$

$$(2\alpha_1\ -\alpha_1\ 0)(\alpha_2 + 2\alpha_2 - \alpha_2) + (0\ 2\alpha_3\ -\alpha_3) = 3\ 2\ 1$$

$$2\alpha_1 + \alpha_2 = 3 \quad -(1)$$
$$-\alpha_1 + 2\alpha_2 + 2\alpha_3 = 2 \quad -(2)$$
$$-\alpha_2 - \alpha_3 = 1 \quad -(3)$$

$$\cancel{(2) + 2(3)}$$
$$\cancel{-\alpha_1 - \alpha_2 = 3} \quad -(4)$$

~~subtract (1) and (4)~~

$$\cancel{-\alpha_1 = 0}$$
$$\cancel{\alpha_1 = 0}$$

~~from (2)~~

$$\cancel{\alpha_3 = 3 - 2\alpha_1}$$

from (1) $\quad 2\alpha_1 + \alpha_2 = 3$

$$2\alpha_1 = 3 - \alpha_2$$
$$\alpha_1 = \frac{3}{2} - \frac{\alpha_2}{2} \quad -(4)$$

from (3) $\quad \rightarrow -\alpha_3 = 1 + \alpha_2$

$$\alpha_3 = -\alpha_2 - 1 \quad -(5)$$

Put in (2)

$$-\left[\frac{3}{2} - \frac{\alpha_2}{2}\right] + 2\alpha_2 + 2[-\alpha_2 - 1] = 2$$

$$-\frac{3}{2} + \frac{\alpha_2}{2} + 2\alpha_2 - 2\alpha_2 + 2 = 2$$

$$\frac{\alpha_2}{2} = 4 + \frac{3}{2}$$
$$\frac{\alpha_2}{2} = \frac{8+3}{2}$$
$$\boxed{\alpha_2 = 11}$$ Put in

find the value of k, such that the column of matrix $\begin{bmatrix} 2 & 1 & k \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ are

(i) L.I

(ii) L.D

Ans i, $k \neq -9$

ii $k = -9$

Find the value of k so that the vectors

$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ -\frac{3}{2} \end{bmatrix}$ and $\begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix}$ are L.D

Ans : $k = \frac{3}{4}$

$2(\alpha_1) + 11 = 3$

$2\alpha_1 = 3 - 11$

$2\alpha_1 = -8$

$\boxed{\alpha_1 = -\frac{8}{2}}$ $\boxed{\alpha_1 = -4}$

Put $\alpha_2 = 11$ in (3)

$-11 - \alpha_3 = 1$ $\Rightarrow$ eqn

$-\alpha_3 = 11 + 1$

$-\alpha_3 = 12$

$\boxed{\alpha_3 = -12}$

$\alpha_2 = 11, \alpha_1 = -8/_2, \alpha_3 = 12$ Ans

Find the value of k, such that the columns of matrix

$$\begin{bmatrix} 2 & 1 & k \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$ are I L.I
II L.D

sol:
$$\begin{bmatrix} 2 & 1 & k \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

I In case of L.I. $|A| \neq 0$

$$2(-2-2) - 1[1-0) + k[-1-0) \neq 0$$

$$-8 - 1 - k \neq 0$$

$$-9 - k \neq 0$$

$$-k \neq 9$$

$$k \neq -9$$

II In case of L.D, $|A| = 0$

$$\cancel{k=9}$$

$$-8 - 1 - k = 0$$

$$-9 - k = 0$$

$$k = -9$$

Find the value of k so that the vector

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix}$$ are L.D

$$\begin{bmatrix} 1 & 1 & k \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

$$1(2-0) - 1[-1) + k[+2-6] = 0$$

$2+1-4k=0$

$3-4k=0$

$3=4k$

$\frac{3}{4}=k$ Ans

Examine whether $(1,-3,5)$ belongs to the linear space generated by S, where $S = \{(1,2,1),(1,1,-1),(4,5,-2)\}$ or not defined?

$$(1,-3,5) = \alpha_1(1,2,1) + \alpha_2(1,1,-1) + \alpha_3(4,5,-2)$$
$$= (\alpha_1 + 2\alpha_1 + \alpha_1) + (\alpha_2 + \alpha_2 - \alpha_2) + (4\alpha_3 + 5\alpha_3 - 2\alpha_3)$$

$$(1,-3,5) = (\alpha_1 + \alpha_2 + 4\alpha_3)(2\alpha_1 + \alpha_2 + 5\alpha_3)(\alpha_1 - \alpha_2 - 2\alpha_3)$$

$\alpha_1 + \alpha_2 + 4\alpha_3 = 1$ — (1)

$2\alpha_1 + \alpha_2 + 5\alpha_3 = -3$ — (2) -3+5

$\alpha_1 - \alpha_2 - 2\alpha_3 = 5$ — (3)

Add (1) and (2)

$2\alpha_1 + 2\alpha_3 = 6$ — (2)

$\alpha_1 + \alpha_3 = 3$ — (4)

Add (2) and (3)

$3\alpha_1 + 3\alpha_3 = 2$ — (5)

$\alpha_1 + \alpha_3 = \frac{2}{3}$

No defined.

Prove that the vectors (1, -2, 3) (1, 1, 0) and (-1, 0, 0) are L.I

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$|A| = (1)(0) + 2(0) + 3(-0+1)$$
$$+3 \neq 0$$

The vector are L.I

Prove that vectors (1, -2, 3), (1, 1, 0) and (-1, 0, 0), (4, 3, 5)

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \\ 4 & 3 & 5 \end{bmatrix}_{4 \times 3}$$

$$|B| = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \neq 0$$

rank A < no. of vector
vector are L.D

[2 1 3 0], (3 5 9 6)

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 5 & 9 & 6 \end{bmatrix}_{2 \times 4}$$

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(B) : $\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 7 \neq 0$

Rank = 2

Rank = no. of vectors

$\therefore$ Vectors are L.I.

Remember

If rank < no. of vectors, then vectors are L.D
If rank = " " " " L.I

(1, 2, 3 4), (1 5 9 6), (1 3 6 4)

$$\cancel{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 5 & 9 & 6 \\ 1 & 3 & 6 & 4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 3 \\ 3 & 9 & 6 \\ 4 & 6 & 4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

eq < 2 variable L.D
$x + y = 0$
$2x + 3y = 0$
no of eq = no. of variables

(1 1 2 3) (1 4 5 6) 3 9 7 2)

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 4 & 9 \\ 2 & 5 & 7 \\ 3 & 6 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Q

Determine whether following matrix have same row space or not? (rank say)

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 3 & 15 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 4 & 3 & -1 \\ 3 & -1 & 3 \end{bmatrix}$$

Sol: $|A| \neq \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \Rightarrow (-2+3) = 1 \neq 0$

Rank = 2

$|B| = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \Rightarrow (3-2) = 1 \neq 0$

Rank = 2

$|C| = \begin{bmatrix} 1 & -1 & -1 \\ 4 & 3 & -1 \\ 3 & -1 & 3 \end{bmatrix} \Rightarrow 1[9-1] + 1[12+3] - 1(-4-9)$

$= 8 + 15 + 13$

$= 36 \neq 0$

Rank = 3

AB are row space and BC ~~or~~ AC are not row space.

under what conditions on a, the vectors $(1-a, 1+a)$ and $(1+a, 1-a)$ are L.D

Sol: $(A) = \begin{bmatrix} 1-a & 1+a \\ 1+a & 1-a \end{bmatrix}$

$|A| = \big((1-a)(1-a) - (1+a)(1+a)\big)$
$= \big(1-a-a+a^2 - (1+a+a+a^2)\big)$
$= \big(\cancel{1} - 2a + \cancel{a^2} - \cancel{1} - 2a - \cancel{a^2}\big)$
$= \cancel{-4a+2}$
$-4a = 0$
$\boxed{a=0}$ Ans

Prove that the vector $(0,1,0), (0,0,1), (1,0,0)$ and $(1,1,1)$, L.D

Sol:- $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{3\times4}$

$|A| = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$ rank = 3

$= 1(1) = 1 \neq 0$

$1 \neq 0$ Ans

(B) If u, v, w are L.I vectors, then show that
(i) $u+v$, $v+w$, and $w+u$ are also L.I

Sol:- Let $\alpha_1(u+v) + \alpha_2(v+w) + \alpha_3(w+u) = 0$
$(\alpha_1 u + \alpha_1 v) + (\alpha_2 v + \alpha_2 w) + (\alpha_3 w + \alpha_3 u) = 0$
$(\alpha_1 + \alpha_3) u + (\alpha_1 + \alpha_2) v + (\alpha_2 + \alpha_3) w = 0$

As u, v and w are L.I

$\alpha_1 + \alpha_2 = 0$
$\alpha_1 + \alpha_3 = 0$
$\alpha_2 + \alpha_3 = 0$

(ii) Show that u+v, u+v, u+2v+w are L.I

Let $\alpha_1(u+v) + \alpha_2(u+v) + \alpha_3(u+2v+w) = 0$

$\alpha_1 u + \alpha_1 v + \alpha_2 u + \alpha_2 v + \alpha_3 u + \alpha_3 2v + \alpha_3 w = 0$

$(\alpha_1 + \alpha_2 + \alpha_3) u = 0$ — (1)
$(\alpha_1 + \alpha_2 - \alpha_3 2) v = 0$ — (2)
$\alpha_3 w = 0$ — (3)

$\alpha_3 = 0$ Put in (1), (2)

$\alpha_1 + \alpha_2 = 0$
$\alpha_1 + \alpha_2 = 0$

Vector space

$S = \{(1,1), (1,0), (0,1), (1,2), (2,1) \ldots\}$
Take the set of all 2-vectors, then it is called 2-vector space.

$(x,y,z) = \alpha(1,1,1) + \beta(1,1,0) + \gamma(1,0,0)$ for some $\alpha, \beta, \gamma \in R$

$(x,y,z) = (\alpha+\beta+\gamma,\ \alpha+\beta+\alpha)$

$\alpha = z \Rightarrow \alpha + \beta = y \Rightarrow \beta = y - z \Rightarrow \alpha + \beta + \gamma = x$

Basis of n-vector space V_n

A set $S \subseteq V_n$ such that

(i) Every element of V_n is a linear combination of elements of set S.

(ii) S is linearly independent.

Basis $B = \{ \ldots \}$

Basis of n vector space has at most n vectors

Q. V_3

$S = \{(2,1,3), (3,1,0) (1,0,2)\}$

whether S is basis for V_3 or not?

Sol:
$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$2(2) - 1(6) + 3(-1)$

$4 - 6 - 3 \neq 0$

So the vectors are linearly Independent

$(x, y, z) = \alpha(2,1,3) + \beta(3,1,0) + \gamma(1,0,2)$

$(x, y, z) = 2\alpha + 3\beta + \gamma, \alpha + \beta, 3\alpha + 2\gamma$

$2\alpha + 3\beta + \gamma = x$ — (1)

$\alpha + \beta = y$ — (2)

$3\alpha + 2\gamma = z$ — (3)

(3) q. $3\alpha+3\beta=3y$

$2\alpha+\alpha+3\beta=3y$

$2\alpha+3\beta=3y-\alpha$

Put this value in eqn (1)

$3y-\alpha-\gamma=x$

$-\alpha+\gamma=x-3y$ — (4)

from (3) $3\alpha+2\gamma=2$

$-\alpha+\gamma=x-3y$

$-3\alpha+3\gamma=3x-9y$

$\underline{\cancel{3\alpha}+2\gamma=2}$

$5\gamma=3x-9y+2$

$\gamma=\frac{3x-9y+2}{5}$ — put in (4)

$-\alpha+\frac{3x+9y+3}{5}=x-3y$

$-\alpha=x-3y-\left(\frac{3x+9y+3}{5}\right)$

$-\alpha=\frac{5x-15y-3x+9y-3}{5}$

$-\alpha=\frac{2x+6y-3}{5}$ $\Rightarrow$ $\alpha=\frac{-2x-6y+3}{5}$

from (2) $\alpha+\beta=y$

$\beta=y+\left(\frac{2x+6y-3}{5}\right)$

$\frac{5y+2x+6y-3}{5}$ $\Rightarrow$ $\frac{2x+11y-3}{5}$

In vector space

then $B=\{(1,0,0\ldots),(0,1,0\ldots),(0,0,1,0\ldots)$

$B=\{e_1,e_2,e_3,e_4\ldots e_n\}$ standard basis of V_n

Subspace of n-vector space : $S \subseteq V_n$ is called subspace of V_n if

(i) if $v_1, v_2 \in S$, then $v_1+v_2 \in S$
(ii) $\alpha \in R$, then $\alpha v_1 \in S$

Example $S \subseteq V_4$

$S = \{(1,2,3,4), (2,4,6,8), (3,6,9,12)\}$

$v_1+v_2 = v_3 \in S$
$v_2+v_3 = (5\ 10\ 15\ 20)$
$\notin S$

$S = \{(1\,1\,1\,1), (2\,2\,2\,2)\,(3\,3\,3\,3), (4\,4\,4\,4), (5,5,5,5) \ldots\}$

Sol $(x\ x\ x\ x) + (y\ y\ y\ y) = (x+y\ x+y\ x+y\ x+y)$
$\in S$

(ii) $\alpha = -1$
$\alpha(x\,x\,x\,x) = (-x\ -x\ -x\ -x) \notin S$

$S = \{\alpha(1,2,3,4) ; \alpha \in IR\}$

$\alpha_1(1\,2\,3\,4) + \alpha_2(1\,2\,3\,4) = (\alpha_1+\alpha_2)(1\,2\,3\,4)$

$v_1 \in S$
$B \in IR$
$Bv_1 = B\alpha_1(1\,2\,3\,4) \in S$,

let $u \in V_4$

$S\{v\}$, then S is independent

$V = S, \{\alpha u ; \alpha \in \mathbb{R}\}$ is a spacepace of V_4
$\langle u \rangle$

$\{u\}$ is independent and every element of S, can be written as linear combination of vector v

$\Rightarrow$ $\{v\}$ is a basis of subspace S, of V_4

Conclusion

1 Basis of a n-vector space has exactly n elements

2. Basis of subspace of a subspace v of vector space V_n has $\leq n$ elements.

write a basis of V_5 other than standard basis
e_1, e_2, e_3, e_4, e_5

$\{e_1+e_2+e_3+e_4+e_5, e_1+e_2+e_3+e_4, e_1+e_2+e_3, e_1+e_2, e_1\}$

Linear Transformation

Let U_m and V_n be two vector space.

Then linear Transformation from U_m to V_n is defined as

$$T : U_m \rightarrow V_n$$

S-T

i.e. $U_m = \{(x_1, x_2 \ldots x_m) \; x_i \in \mathbb{R}\}$
$V_n = \{(x_1 \ldots x_n) \; x_i \in \mathbb{R}\}$

S-T for all $u_1, u_2 \in U_m$.

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

(ii) $T(\alpha u_1) = \alpha T(u_1) \;\; \forall \alpha \in \mathbb{R}$

OR for all $u_1, u_2 \in U_m$

(i) $T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2) \;\; \forall \alpha, \beta \in \mathbb{R}$

Prove that $T: V_3 \rightarrow V_2$ defined by $T(a, b, c) = (a, b) \; \forall (a, b, c) \in V_3$ is a linear transformation.

Sol! 1st way

Let $u_1, u_2 \in V_3$
$u_1 = (a_1, b_1, c_1)$, $u_2 = (a_2, b_2, c_2)$ where $a_i, b_i \in \mathbb{R} \; \forall i, j$

~~Then $u_1 (a_1, b_1, c_1)$, $u_2 = (a_2, b_2, c_2)$ where~~
take $\alpha, \beta \in \mathbb{R}$

$T(\alpha v_1 + \beta v_2) = T\left((\alpha a_1, \alpha b_1, \alpha c_1) + \beta a_2 + \beta b_2 + \beta c_2\right)$

$= T\left((\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)\right)$

$= (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2)$.

$= \alpha(a_1, b_1) + \beta(a_2, b_2)$ — (1)

Let

$\alpha T(v_1) + \beta T(v_2) = \alpha T((a_1, b_1, c_1)) + \beta T((a_2, b_2, c_2))$

$= \alpha(a_1, b_1) + \beta(a_2, b_2)$ — (2)

from (1) and (2)

$T(\alpha v_1 + \beta v_2) =$

$\alpha T(v_1) + \beta T(v_2) \; \forall v_1, v_2 \in V_3$ and $\forall \alpha, \beta \in \mathbb{R}$.

check whether $T: V_2 \to V_2$ defined by $T(a, b) = (1 + a, b)$ is L.T or not

Then $v_1 = (a_1, b_1)$, $v_2 = (a_2, b_2)$ where $a_i, b_j \in \mathbb{R}$ $\forall i, j$

take $\alpha, \beta \in \mathbb{R}$

$\therefore T(\alpha v_1 + \beta v_2) = T((\alpha a_1, \alpha b_1) + (\beta a_2, \beta b_2))$

$= T\left((\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2)\right)$

$= (1 + \alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2)$ — (1)

$\alpha T(u_1)+\beta T(u_2) = \alpha T(a_1 b_1) + \beta T(a_2, b_2))$

$= \alpha(1+a_1, b_1) + \beta(1+a_2, b_2)$.

$= \alpha + \alpha a_1 + \beta + \beta a_2 + \beta b_2$ — (2)

from (1) and (2)

$T(\alpha u_1 + \beta u_2) \neq \alpha T(u_1) + \beta T(u_2)$
$\forall u_1, u_2 \in V_3$ and $\forall \alpha, \beta \in \mathbb{R}$

check whether the following are L.T or not?

(i) $T: V_2 \to V_2$ defined by $T(a,b) = (b,a)$
(ii) $T: V_2 \to V_3$ defined by $T(a,b) = (a-b, b-a, -a)$

(i) Let u_1, u_2
$u_1 = a_1 b_1$ and $u_2 = a_2, b_2$

$T(\alpha u_1 + \beta u_2) = T((\alpha a_1, \alpha b_1) + (\beta a_2, \beta b_2))$

$= T((\alpha a_1 + \beta a_2), (\alpha b_1 + \beta b_2))$
$= \alpha(a_1, b_1) + \beta(a_2, b_2)$ — (2)

$\alpha T(u_1) + \beta T(u_2) = \alpha T(a_1 b_1) + \beta T(a_2 b_2)$

$= (\beta a_2, \beta b_2) + (\alpha a_1, \alpha b_1)$
$= (\beta a_2 + \alpha a_1), (\beta b_2 + \alpha b_1)$
$= \beta(a_2 + b_2) + \alpha(a_1 + b_1)$ — (1)

$\alpha T(u_1) + \beta T(u_2) = \alpha T(a_1, b_1) + \beta T(a_2, b_2)$

$= \alpha(a_1, b_1) + \beta(a_2, b_2)$

$= (\alpha a_1, \alpha b_1) + (\beta a_2, \beta b_2)$

$= (\alpha a_1 + \beta a_2), (\alpha b_1 + \beta b_2)$

$= \beta(a_2 + b_2), \alpha(a_1 + b_1)$ — (2)

from (1) and (2)

$T(\alpha u_1 + \alpha u_2) = \alpha T(u_1) + \beta T(u_2)$ Ans

Subspace.

(iii) $T: V_3 \to V_3$ defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$

(iv) $T: V_4 \to V_3$ defined by $T(a, b, c, d) = (a - b + c + d, a + 2c - \ldots, a + b + 3c - 3d)$

yes (v) $T: V_3 \to V_2$ defined as $T((a, b, c)) = (a - b, a - c)$

(vi) $T: V_2 \to V_2$ defined as $T(a, b) = (b, a)$

No (vii) $T: V_2 \to V_3$ defined by $T(a, b) = (b, a, 1)$

yes (viii) $T: V_2 \to V_4$ defined by $T(a, b) = (a, b, 0, 0)$

(ii). $T: V_2 \to V_3$ defined by $T(a, b) = (a - b, b - a, -a)$

Sol:- let $u_1, u_2 \in V_3$

~~$u_1 = a_1 b_1 c_1$ and $u_2 = a_2 b_2 c_2$~~

~~$T(\alpha u_1 + \beta u_2) = T(\alpha(a_1 b_1 c_1) + \beta(a_2 b_2 c_2))$~~

~~$= T(\alpha a_1, \alpha b_1, \alpha c_1 + \beta a_2, \beta b_2, \beta c_2)$~~

~~$= T($~~

Let $v_1, v_2 \in V_3$
$v_1 = a_1 b_1 \quad v_2 = a_2 b_2$

$T(\alpha v_1 + \beta v_2) = T(\alpha(a_1 b_1) + \beta(a_2 b_2))$

$= T(\alpha a_1, \alpha b_1 + \beta a_2, \beta b_2)$
$= T(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2)$
$= (\alpha a_1 + \beta a_2 - \alpha b_1 - \beta b_2, \alpha b_1 + \beta b_2 - \alpha a_1 - \beta a_2, -\alpha a_1 - \beta a_2)$

$= (\alpha a_1 - \alpha b_1 + \beta a_2 - \beta b_2, \alpha b_1 - \alpha a_1 + \beta b_2 - \beta a_2, -\alpha a_1 - \beta a_2)$ — (1)

$\alpha T(v_1) + \beta T(v_2) = \alpha T(a_1, b_1) + \beta T(a_2, b_2)$

$= \alpha(a_1 - b_1, b_1 - a_1, -a_1) + \beta(a_2 - b_2, b_2 - a_2, -a_2)$

$= (\alpha a_1 - \alpha b_1, \alpha b_1 - \alpha a_1, -\alpha a_1) + (\beta a_2 - \beta b_2, \beta b_2 - \beta a_2, -\beta a_2)$

$= (\alpha a_1 - \alpha b_1 + \beta a_2 - \beta b_2, \alpha b_1 - \alpha a_1 + \beta b_2 - \beta a_2, -\alpha a_1 - \beta a_2)$ — (2)

from (1) & (2)

$T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2) \; \forall \; v_1, v_2 \in V_3$
$\forall \; \alpha, \beta \in \mathbb{R}$

(ii) $T: V_3 \to V_3$ defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$

Sol:- $T(\alpha v_1 + \beta v_2 + \gamma v_3) = T(\alpha(a_1 b_1 c_1) + \beta(a_2 b_2 c_2) + \gamma(a_3 b_3 c_3))$

$= T(\alpha a_1, \alpha b_1, \alpha c_1 + \beta a_2, \beta b_2, \beta c_2 + \gamma a_3, \gamma b_3, \gamma c_3)$
$= (\alpha a_1, \alpha b_1, \alpha c_1 + 2(\beta a_2, \beta b_2, \beta c_2) - (\gamma a_3, \gamma b_3, \gamma c_3), (\beta a_2, \beta b_2, \beta c_2$
$+ \gamma a_3, \gamma b_3, \gamma c_3), (\alpha a_1, \alpha b_1, \alpha c_1 + \beta a_2, \beta b_2, \beta c_2 - 2(\gamma a_3, \gamma b_3, \gamma c_3)$

$x+2y=3,\ y+z,\ x+y=2z$

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R.H.S $= \alpha T(u_1) + \beta T(u_2) + \gamma T(u_3)$

$\alpha T(a_1, b_1, c_1) + \beta T(a_2, b_2, c_2) + \gamma T(a_3, b_3, c_3)$

$T(\alpha a_1, \alpha b_1, \alpha c_1) + T(\beta a_2, \beta b_2, \beta c_2) + T(\gamma a_3, \gamma b_3, \gamma c_3)$

$= (\alpha a_1 + 2\alpha b_1 - \alpha c_1, \alpha b_1 + \alpha c_1, \alpha a_1 + \alpha b_1 - 2\alpha c_1) + (\beta a_2 + 2\beta b_2 - \beta c_2, \beta b_2 + \beta c_2, \beta a_2 + \beta b_2 - 2\beta c_2) + (\cancel{\gamma a_3 + \gamma b_3 - 2} \gamma a_3 + 2\gamma b_3 - \gamma c_3, \gamma b_3 + \gamma c_3, \gamma a_3 + \gamma b_3 - 2\gamma c_3)$

$= (\alpha a_1 + 2\alpha b_1 - \alpha c_1 + (\beta a_2 + 2\beta b_2 - \beta c_2) + \gamma a_3 + 2\gamma b_3 - \gamma c_3),$
$(\alpha b_1 + \alpha c_1 + \beta b_2 + \beta c_2 + \gamma b_3 + \gamma c_3),$
$(\alpha a_1 + \alpha b_1 - 2\alpha c_1 + \beta a_2 + \beta b_2 - 2\beta c_2 + \gamma a_3 + \gamma b_3 - 2\gamma c_3)$

$= \alpha a_1 + 2\alpha b_1 - \alpha c_1 + \beta a_2 + 2\beta b_2 - \beta c_2 + \gamma a_3 + 2\gamma b_3 - \gamma c_3,$
$\alpha b_1 + \alpha c_1 + \beta b_2 + \beta c_2 + \gamma b_3 + \gamma c_3,$
$\alpha a_1 + \alpha b_1 - 2\alpha c_1 + \beta a_2 + \beta b_2 - 2\beta c_2 + \gamma a_3 + \gamma b_3 - 2\gamma c_3$

~~T: $V_4 \to V_3$ defined by $T(a, b, c, d) = (a - b + c + d, a + 2c - d, a + b + 3c - 3d)$~~

Matrix of a linear transformation

$T: V_2 \to V_3$

$T(x,y) = (x,y,0)$ $\qquad B_2 = \{(1,0),(0,1)\}$

$T(1,0) = (1,0,0)$
$T(0,1) = (0,1,0)$

$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3\times 2}$ - matrix of linear transforma

check whether $T: V_3 \to V_2$
$T(x,y,z) = (x+y, y+z, \cancel{\ })$
is a L.T and find of its matrix

Sol:- $u_1 = (a_1 b_1 c_1)$, $u_2 = (a_2 b_2 c_2)$

LHS $T(\alpha(u_1) + \beta u_2) = T(\alpha(a_1 b_1 c_1) + \beta(a_2 b_2 c_2))$

$= T(\alpha a_1, \alpha b_1, \alpha c_1 + \beta a_2, \beta b_2, \beta c_2)$

$= T[(\alpha a_1 + \beta a_2), (\alpha b_1 + \beta b_2), (\alpha c_1 + \beta c_2)]$

$= (\alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2), (\alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2)$

R.H.S $T\alpha(u_1) + T\beta(u_2) \Rightarrow T\alpha(a_1 b_1 c_1) + T\beta(a_2 b_2 c_2)$

$= T(\alpha a_1, \alpha b_1, \alpha c_1) + T(\beta a_2, \beta b_2, \beta c_2)$

$= (\alpha a_1 + \alpha b_1, \alpha b_1 + \alpha c_1) + (\beta a_2 + \beta b_2, \beta b_2 + \beta c_2)$

$= (\alpha a_1 + \alpha b_1 + \beta a_2 + \beta b_2), (\alpha b_1 + \alpha c_1 + \beta b_2 + \beta c_2)$

L.H.S = R.H.S
Hence it is linear transformation

$T(e_1) = T(1,0,0) \Rightarrow (1,0)$
$T(e_2) = T(0\ 1\ 0) \Rightarrow (0,1)$
$T(e_3) = T(0\ 0\ 1) = (0,0)$

$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \quad T = \begin{array}{c} 1 \\ 0 \\ 0 \end{array}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ Ans

find the matrix of following linear transformation.

1) $T: V_2 \to V_2$
$T(x,y) = (4x-2y, 2x+y)$

ii) $T: V_3 \to V_3$

$T(x,y,z) = (x+y+z, -x-y-4z, 2x-z)$

iii) $T: V_2 \to V_2$
$T(x,y) = (x+y, x+y)$

4) $T: V_2 \to V_2$
$T(a,b) = (a,0)$

5) $T: V_5 \to V_5$
$T(x_1, x_2, x_3, x_4, x_5) = (x_5, x_4, x_3, x_2, x_1)$ 5x5

6) $T: V_3 \to V_1$
$T(a,b,c) = a-2b+c$
1x3

(7) $T: V_3 \to V$

$T(a,b,c) = c+1$

(8) $T: V_3 \to V$

$T(a,b,c) = c+1$

Solution

(1) $T: V_2 \to V_2$

$T(x,y) = (4x-2y,\ 2x+y)$

Let $u_1, u_2 \in V_2$

$u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2)$

L.H.S: $T(\alpha u_1 + \beta u_2) = T(\alpha(a_1,b_1) + \beta(a_2,b_2))$

$= T(\alpha a_1, \alpha b_1 + \beta a_2, \beta b_2)$

$= \cancel{4(\alpha a_1, \alpha b_1)}$

$= T(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2)$

$= (4(\alpha a_1 + \beta a_2) - 2(\alpha b_1 + \beta b_2),\ 2(\alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2)$

$= (4\alpha a_1 + 4\beta a_2 - 2\alpha b_1 + 2\beta b_2,\ 2\alpha a_1 + 2\beta a_2 + \alpha b_1 + \beta b_2)$

R.H.S $T\alpha(u_1) + T\beta(u_2) = T\alpha(a_1,b_1) + T\beta(a_2,b_2)$

$= T(\alpha a_1, \alpha b_1) + T(\beta a_2, \beta b_2)$

$= (4\alpha a_1 - 2\alpha b_1,\ 2\alpha a_1 + \alpha b_1) + (4\beta a_2 - 2\beta b_2,\ 2\beta a_2 + \beta b_2)$

$= 4\alpha a_1 - 2\alpha b_1 + 4\beta a_2 - 2\beta b_2,\ 2\alpha a_1 + \alpha b_1 + 2\beta a_2 + \beta b_2$

L.H.S = R.H.S

$(4, x-2y, 2x+y)$

Hence T is linear transformation.

$T(e_1) = (1,0) = \cancel{(4,0)}, \cancel{(2,0)} \; (4, 2)$

$T(e_2) = (0,1) = \cancel{0} \; (-2, 1)$

$$\begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}$$

matrix of linear transformation.

11) $T: V_3 \rightarrow V_3$

$T(x,y,z) = (x+y+z, -x-y-4z, 2x-z)$

$u_1 = (a_1, b_1, c_1), \quad u_2 = (a_2, b_2, c_2)$

LHS: $T(\alpha u_1 + \beta u_2) = T(\alpha(a_1, b_1, c_1) + \beta(a_2, b_2, c_2))$

$= T[(\alpha a_1, \alpha b_1, \alpha c_1) + (\beta a_2, \beta b_2, \beta c_2)]$

$= T(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$

$= (\alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2, -\alpha a_1 - \beta a_2 - \alpha b_1 - \beta b_2 - 4\alpha c_1 - 4\beta c_2, 2\alpha a_1 + 2\beta a_2 - \alpha c_1 - \beta c_2)$

$T\alpha u_1 + T\beta u_2 = T\alpha(a_1 b_1 c_1) + T\beta(a_2 b_2 c_2)$

$= T(\alpha a_1, \alpha b_1, \alpha c_1) + T(\beta a_2, \beta b_2, \beta c_2)$

$= (\alpha a_1 + \alpha b_1 + \alpha c_1, -\alpha a_1 - \alpha b_1 - 4\alpha c_1, 2\alpha a_1 - \alpha c_1) + (\beta a_2 + \beta b_2 + \beta c_2, -\beta a_2 - \beta b_2 - 4\beta c_2, 2\beta a_2 - \beta c_2)$

$= \alpha a_1 + \alpha b_1 + \alpha c_1 + \beta a_2 + \beta b_2 + \beta c_2, -\alpha a_1 - \alpha b_1 - 4\alpha c_1 - \beta a_2 - \beta b_2 - 4\beta c_2, 2\alpha a_1 - \alpha c_1 + 2\beta a_2 - \beta c_2$

$(0\ 1\ 0)\ (1\ -1\ 0)\ (2\ 0)$

L.H.S = R.H.S
Hence it is linear transformation

$T(e_1) = (1,0,0) \quad = (1,-1,2)$
$T(e_2) = (0,1,0) \quad = (1\ -1\ 0)$
$T(e_3) = (0,0,1) \quad (1\ -4\ -1)$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -4 \\ 2 & 0 & -1 \end{bmatrix}$$ matrix of linear transformation.

IV) $T: V_2 \to V_2$
$T(a,b) = (a,0)$
Sol:- $u_1 = (a_1, b_1)$, $u_2 = (a_2 b_2)$

LHS. $T(\alpha u_1 + \beta u_2) = T(\alpha(a_1 b_1) + \beta(a_2 b_2))$
$= T(\alpha a_1, \alpha b_1 + \beta a_2, \beta b_2)$
$= T(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2)$
$= (\alpha a_1 + \beta a_2, 0)$

RHS $T\alpha(u_1) + T\beta(u_2) = T\alpha(a_1 b_1) + T\beta(a_2 b_2)$
$= T(\alpha a_1 \alpha b_1) + T(\beta a_2 \beta b_2)$
$= (\alpha a_1\ 0) + (\beta a_2, 0)$
$= (\alpha a_1 + \beta a_2, 0)$

Hence it is linear transformation

$T(e_1) = (1,0) \quad (1,0)$
$T(e_2) = (0\,1) \quad (0\ 0)$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$ Ans

V $T: V_5 \to V_5$
$T(x_1, x_2, x_3, x_4, x_5) = (x_5, x_4, x_3, x_2, x_1)$
Sol: $u_1 = (a_1 b_1 c_1 d_1 e_1)$, $u_2 = (a_2 b_2 c_2 d_2 e_2)$
LHS:- $T(\alpha u_1 + \beta u_2) = T\alpha(a_1 b_1 c_1 d_1 e_1) + \beta(a_2 b_2 c_2 d_2 e_2)$
$= T(\alpha a_1, \alpha b_1, \alpha c_1, \alpha d_1, \alpha e_1) + \beta a_2, \beta b_2, \beta c_2, \beta d_2, \beta e_2)$
$= T(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2, \alpha d_1 + \beta d_2, \alpha e_1 + \beta e_2)$

$= (\alpha e_1 + \beta e_2), (\alpha d_1 + \beta d_1), (\alpha c_1 + \beta c_1), (\alpha b_1 + \beta b_2),$
$(\alpha a_1, \beta a_2)$

R.H.S. $T\alpha(u_1) + T\beta(u_2) = T\alpha(a_1, b_1, c_1, d_1, e_1) + T\beta(a_2, b_2, c_2, d_2, e_2)$

$= T(\alpha a_1, \alpha b_1, \alpha c_1, \alpha d_1, \alpha e_1) + T(\beta a_2, \beta b_2, \beta c_2, \beta d_2, \beta e_2)$

$= \alpha e_1, \alpha d_1, \alpha c_1, \alpha b_1, \alpha a_1 + \beta e_2, \beta d_2, \beta c_2, \beta b_2, \beta a_2$

$= (\alpha e_1 + \beta e_2), (\alpha d_1 + \beta d_2), (\alpha c_1 + \beta c_2), (\alpha b_1 + \beta b_2),$
$(\alpha a_1 + \beta a_2)$

L.H.S = R.H.S

$T(e_1) = (1,0,0,0,0)$ $(0,0,0,0,1)$
$T(e_2) = (0,1,0,0,0)$ $(0\ 0\ 0\ 1\ 0)$
$T(e_3) = (0,0,1,0,0)$ $(0\ 0\ 1\ 0\ 0)$
$T(e_4) = (0,0,0,1,0)$ $(0\ 1\ 0\ 0\ 0)$
$T(e_5) = (0,0,0,0,1)$ $(1\ 0\ 0\ 0\ 0)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Ans

(6) $T: V_3 \to V_1$
$T(a, b, c) = a - 2b + c$

$u_1 = (x_1, y_1, z_1)$, $u_2 = (x_2, y_2, z_2)$

$T(\alpha u_1 + \beta u_2) = T[\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)]$
$= T(\alpha x_1, \alpha y_1, \alpha z_1) + \beta x_2, \beta y_2, \beta z_2$
$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$
$= \alpha x_1$
$= \alpha x_1 - 2\alpha y_1 + \alpha z_1$

$T(\alpha u_1 + \beta u_2) = T(\alpha(x_1 y_1 z_1) + \beta(x_2 y_2 z_2))$
$= T((\alpha x_1, \alpha y_1, \alpha z_1) + \beta x_2, \beta y_2, \beta z_2)$
$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$
$= \alpha x_1 + \beta x_2 - 2\alpha y_1 \, 2\beta y_2 + \alpha z_1 + \beta z_2$

$T\alpha(u_1) + T\beta(u_2) = T\alpha(x_1, y_1, z_1) + \beta(x_2 y_2, z_2)$
$= T(\alpha x_1, \alpha y_1, \alpha z_1) + \beta x_2, \beta y_2, \beta z_2)$
$= \alpha x_1 - 2\alpha y_1 + \alpha z_1 + \beta x_2 - 2\beta y_2 + \beta z_2$
$= \alpha x_1 + \beta x_2, -2\alpha y_1 - 2\beta y_2 + \alpha z_1 + \beta z_2$

L.H.S = R.H.S

$T(e_1) = (1, 0, 0) \Rightarrow (1)$
$T(e_2) = (0, 1, 0) \quad (-2)$
$T(e_3) = (0, 0, 1) \quad 1$

$(1 \; -2 \; 1)$ Ans

(7) $T: V_3 \to V$
$T(a, b, c) = c + 1$

$u_1 = (x_1 y_1 z_1), \; u_2 = (x_2, y_2, z_2)$

$T(\alpha u_1 + \beta u_2) = T(\alpha(x_1 y_1 z_1) + \beta(x_2 y_2 z_2))$
$= T((\alpha x_1, \alpha y_1; \alpha z_1) + \beta x_2, \beta y_2, \beta z_2)$

$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$
$= \alpha z_1 + \beta z_2 + 1$

$T\alpha(u_1) + T\beta(u_2) = T\alpha(x_1 y_1 z_1) + \beta T(x_2 y_2 z_2)$
$= T(\alpha x_1, \alpha y_1, \alpha z_1) + T(\beta x_2, \beta y_2, \beta z_2)$
$= \alpha z_1 + 1 + \beta z_2 + 1$
$= \alpha z_1 + \beta z_2 + \cancel{2}$

Unit - 3 linear equations

$$2x+3y+4z=10$$
$$x+2y+3z=14$$
$$x+4y+7z=10$$

$$A=\begin{bmatrix}2&3&4\\1&2&3\\1&4&7\end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix}1&2&3\\2&3&4\\1&4&7\end{bmatrix} \sim \begin{bmatrix}1&2&3\\0&-1&-2\\0&2&4\end{bmatrix} \quad \begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3-R_1\end{matrix}$$

$$\sim\begin{bmatrix}1&2&3\\0&-1&-2\\0&0&0\end{bmatrix} \quad R_3\to R_3+2R_2$$

$$\sim\begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix} \qquad \rho(A)=2$$

$$A:B=\begin{bmatrix}2&3&4&:&10\\1&2&3&:&14\\1&4&7&:&10\end{bmatrix}$$

Row operation

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 14 \\ 2 & 3 & 4 & 10 \\ 1 & 4 & 7 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 14 \\ 0 & -1 & -2 & -18 \\ 0 & 2 & 4 & -4 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 14 \\ 0 & -1 & -2 & -18 \\ 0 & 0 & 0 & -40 \end{array}\right]$$

$R_2 \to -R_2$, $R_3 \to \frac{1}{-40} R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 14 \\ 0 & 1 & 2 & 18 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\rho(A:B) = 3 \neq \rho(A)$

Ranks are not same.

#

$x+y+z = 9$

$2x+5y+7z = 52$

$2x+y-z = 0$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -\frac{4}{3} \end{bmatrix} \quad R_3 \to R_3 + \frac{1}{3} R_2$$

$R_2 \to \frac{1}{3}R_2$, $R_3 \to -\frac{3}{4}R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5/3 \\ 0 & 0 & 1 \end{bmatrix} \quad \rho(A) = 3$$

$\rho(A:B)$

$$A:B = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{array}\right] \quad R_2 \to R_2 - 2R_1 , R_3 \to R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4/3 & -20/3 \end{array}\right] \quad R_3 \to R_3 + \frac{1}{3}R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & 1 & 5/3 & 34/3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$\rho(A:B) = 3$

$\rho(A:B) = 3 = \rho(A)$

So, The system of equation has solution

AX

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 9 \\ 34/3 \\ 5 \end{bmatrix}$$

$\boxed{z = 5}$

$y + \frac{5}{3} z = \frac{34}{3}$

$y + \frac{5}{3}(5) = \frac{34}{3} \Rightarrow y + \frac{25}{3} = \frac{34}{3}$

$y = \frac{34}{3} - \frac{25}{3} \Rightarrow y = \frac{9}{3} 3$

$\boxed{y = 3}$

$x + y + z = 9$

$x + 3 + 5 = 9$

$x + 8 = 9$

$\boxed{x = 1}$

$\boxed{x = 1,\ y = 3,\ z = 5}$ Ans

consider, the ~~equation~~ system of eq^ns $AX = B$

$A_{m \times n}\, X_{n \times 1} = B_{m \times 1}$ [m - equation, n - variable]

(i) If $\rho([A:B]) \neq \rho(A)$, then the system of equation has no solution and we say that system of equations is inconsistent

(ii) If $\rho(A:B) = \rho(A) = r$ (say)

then if $r = n$, then the system has a unique solution

if $r < n$, then the system has infinitely many solutions.

In both these cases, we say that the system of equations is consistent.

\#
$$x+y+z=4$$
$$2x+5y-2z=3$$
$$x+7y-7z=-6$$

Prove that the system of eq^n's is consistent

Sol:- $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$, $A:B = \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 2 & 5 & -2 & : & 3 \\ 1 & 7 & -7 & : & -6 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 6 & -8 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2$$

$$A:B = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & -6 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & -10 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$R_2 \to \frac{1}{3} R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 1 & -4/3 & 5/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$y - 4/3\, z = 5/3$

$y = \frac{4}{3} z - 5/3$, let $z = k$

$$\boxed{y = \frac{4}{3} k - \frac{5}{3}}$$

$x + y + z = 4$

$x + \left(\frac{4}{3}k - \frac{5}{3}\right) + k = 4 \Rightarrow x + \frac{7}{3}k - \frac{5}{3} = 4$

$x = 4 + \frac{5}{3} - \frac{7k}{3} = \boxed{\frac{17}{3} - \frac{7}{3} k}$

∴, it has infinite solution.

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check whether following system of eqⁿ's are consistent or not also solve of the system is consistent.

1. $3x+y-z=1$
$5x+2y+3z=2$
$3y+8x+2z=3$

2. $x-y+2z=4$
$3x+y+4z=6$
$x+y+z=1$

Sol:(1) $A=\begin{bmatrix}3 & 1 & -1\\5 & 2 & 3\\3 & 8 & 2\end{bmatrix}$, $B=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $A:B=\left[\begin{array}{ccc:c}3 & 1 & -1 & 1\\5 & 2 & 3 & 2\\3 & 8 & 2 & 3\end{array}\right]$

$A=\begin{bmatrix}3 & 1 & -1\\5 & 2 & 3\\3 & 8 & 2\end{bmatrix}$

$A=\begin{bmatrix}0 & -7 & 3\\5 & 2 & 3\\3 & 8 & 2\end{bmatrix}$ $R_1 \to R_1 - R_3$

$A=\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$ R

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$x - y + 2z = 4$
$3x + y + 4z = 6$
$x + y + z = 1$

$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$, $A:B = \left[\begin{array}{ccc:c} 1 & -1 & 2 & 4 \\ 3 & 1 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{array}\right]$

$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 2 & -1 \end{bmatrix}$ $\quad R_2 \to R_2 - 3R_1$
$\quad R_3 \to R_3 - R_1$

A.

Trivial (all the values are zero, $x=y=z=0$

non-Trivial (at least one value is zero)

$A'X=B$

$A_{m \times n}(X_{n \times 1}) = B_{m \times 1}$

m equations in n variables

$\rho(A) = \rho(A:B) = n$ unique sol^n

$\rho(A) = \rho(A:B) < n$ infinite sol^n

$\rho(A) \neq \rho(A:B)$ no sol^n

$AX=0$ homogeneous system of eq^n

If $AX \neq 0$ non-homogeneous system of eq^n.

If $B=0$ homogeneous system of equation

Non-homogeneous system of eq^n is $AX=B$, $B \neq 0$

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$x+2y+3z=0$
$3x+4y+4z=0$
$7x+10y+12z=0$
check whether the above system of eqⁿs has a non-zero solⁿ or not.
find $\rho(A)$
no. of variable = 3
if $\rho(A)=3$ then no non-zero sol.
$\rho(A)<3$ then, there are infinity many solⁿ and so a non-zero solution

$$A=\begin{bmatrix}1 & 2 & 3\\ 3 & 4 & 4\\ 7 & 10 & 12\end{bmatrix}$$

4-6
4-6

$$A\sim\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & -2\\ 0 & -4 & -9\end{bmatrix}$$ $R_2\to R_2-3R_1$, $R_3\to R_3-7R_1$

12-21

$$\sim\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & -2\\ 0 & -1 & -\frac{9}{4}\end{bmatrix}$$ ~~$R_3\to R_1-R_3$~~ $R_3=\frac{1}{4}R_3$

$$\sim\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & -2\\ 0 & 1 & 9/4\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & -1\\ 0 & 1 & 9/4\end{bmatrix}$$ $R_2=\frac{1}{2}R_2$

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & 1\\ 0 & 0 & 3/2\end{bmatrix}$$ $R_3\to 2R_3-R_1$

$\frac{18}{4}-5$. $\frac{18-12}{4}=\frac{3}{2}$

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$\rho(A) = 3$		$x+2y+3z=0$
$-7z=0$ $z=0$	$-2y-5z=0$ $-2y=0$ $y=0$	$x=0$ $(0,0,0)$

$x-2y-3z=0$ check whether the above system of eq^n's.

$-2x+3y+5z=0$

$3x+y-2z=0$

$$A = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 3 & 5 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -1 \\ 0 & 7 & 7 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$3, -2, -4$

$1 +$

$$\sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & 7 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to \frac{1}{7} R_3 \\ R_3 \to R_3 - 7R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 < 3$

So the system has infinitely many sol^n

$y+z=0$

$y=-z$

$x+2y-3z=0$

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$x+2z-3z=0$

$x-z=0$

$x=z$

let $z=k$

(k, k, k) are all the sol^n non-zero sol^n can be obtained by taking $k=0$

$x+2y-2z+2s-t=0$

$x+2y-z+3s-2t=0$

$2x+4y-7z+s+t=0$

$$A=\begin{bmatrix}1 & 2 & -2 & 2 & -1\\ 1 & 2 & -1 & 3 & -2\\ 2 & 4 & -7 & 1 & 1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & -2 & 2 & -1\\ 0 & 0 & 1 & 1 & -1\\ 0 & 0 & 3 & -3 & 3\end{bmatrix}\quad \begin{matrix}R_2\to R_2-R_1\\ R_3\to R_3-2R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 2 & -2 & 2 & -1\\ 0 & 0 & 1 & 1 & -1\\ 0 & 0 & 0 & -6 & 6\end{bmatrix}\quad R_3\to R_3-3R_2$$

$-3+7$
$3+7$
$-3-3 \quad 3+3$

$$\sim\begin{bmatrix}1 & 2 & -2 & 2 & -1\\ 0 & 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 0 & 12\end{bmatrix}\quad R_3\to R_3+6R_2$$

$$\sim\begin{bmatrix}1 & 2 & -2 & 2 & -1\\ 0 & 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\quad \begin{matrix}\cancel{R_3\to R_3+12R_1}\\ R_3=\frac{1}{12}R_3\end{matrix}$$

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$$\sim \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x+2y-2z+2s-t=0$ — (1)

$z+s-t=0$ — (2)

from (2)

$t = s+z$ — put in (1) — (4)

$x+2y-2z+2s-(s+z)=0$

$x+2y-2z+2s-s-z=0$

$x+2y-3z+s=0$

$x = -2y+3z-s$ — (3) ans

let $z=0,\ y=0,\ s=0$

~~where solution~~

$x = -0+0-0$

$x=0$ — put in (4)

$t=0$

$(0,0,0,0,0)$

if $y=a,\ z=b,\ s=c$

$x = -2a+3b-c$ — put in (4)

$t = c+b$

$(-2a+3b-c,\ a,\ b,\ c,\ c+b)$

where $a,b,c \in \mathbb{R}$ Ans

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Q. Show that the only real value of k for which the following eq^n's have a non-zero sol^n is 6.

Sol:- $x+2y+3z=kx$
$3x+y+2z=ky$
$2x+3y+z=kz$

$\Rightarrow (1-k)x+2y+3z=0$
$\Rightarrow$ ~~3x + 2~~ $3x+(1-k)y+2z=0$
$\Rightarrow 2x+3y+(1-k)z=0$

$|A|=0$

$$A \begin{vmatrix} 1-k & 2 & 3 \\ 3 & 1-k & 2 \\ 2 & 3 & 1-k \end{vmatrix} = 0$$

$(1-k)[(1-k)(1-k)-6)-2[(1-k)3-4]+3[9-(1-k)2]=0$

$(1-k)[1-k-k+k^2-6]-2[3-3k-4]+3[9-2+2k)=0$

$(1-k)[1-2k+k^2-6]-2[-3k-1]+3[7+2k)=0$

$1-2k+k^2-6-(k-2k^2+k^3-6k)+6k+2+21+6k=0$

$1-2k+k^2-6-k+2k^2-k^3+6k+6k+2+21+6k=0$

$-k^3+3k^2+15k+18=0$ $\qquad k-6\overline{)k^3-3k^2-15k-18}$

$k^3-3k^2-15k-18=0$

$(k-6)(k^2+3k+3)=0$

$k=6,\ k^2+3k+3=0$

$k=\dfrac{-3\pm\sqrt{9-12}}{2},\ \dfrac{-3\pm\sqrt{-3}}{2},\ \dfrac{-3\pm\sqrt{3}\,i}{2}$ Ans

So it is proved that value is 6.

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Find the value of K so that the equation

$x-2y+z=0$
$3x-y+2z=0$ have
$y+Kz=0$

(i) unique solution
(2) Infinity many solution. Also find sol. for these value of K

Ans: (i) $K \neq -\frac{1}{5}$
(ii) $K = -\frac{1}{5}$

Sol:- $A\begin{bmatrix}1 & -2 & 1\\3 & -1 & 2\\0 & 1 & K\end{bmatrix} \neq 0$

$A = \begin{bmatrix}1 & -2 & 1\\0 & 5 & -1\\0 & 1 & K\end{bmatrix}$ $R_2 \to R_2 - 3R_1$

$5K + 1 \neq 0$

$K \neq -\frac{1}{5}$

for infinite $K = -\frac{1}{5}$
sol^n for this value of K

(i) (0,0,0)

(ii) $x-2y+z=0$
$y-z=0$
$y+K=0$

Put $z = a$
$y.$

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Find the value of k such that the system of equation:

$$x + ky + 3z = 0$$
$$4x + 3y + kz = 0$$
$$2x + y + 2z = 0$$ has a non-trivial solⁿ.

Sol: $A = \begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix}$

$|A| = 0$

$$\begin{vmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{vmatrix} = 0$$

$1(6-k) - k(8-2k) + 3(4-6) = 0$

$\cancel{6} - k - 8k + 2k^2 - \cancel{6} = 0$

$2k^2 - 9k = 0$

$k(2k-9) = 0$

$2k - 9 = 0 \Rightarrow 2k = 9 \Rightarrow k = 9/2$

$k = 0, 9/2$

Discuss for all values of k, the system of equations

$$2x + 3ky + (3k+4)z = 0$$
$$x + (k+4)y + (4k+2)z = 0$$
$$x + 2(k+1)y + (3k+4)z = 0$$

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Sol:- $|A| = 0$

$$\begin{vmatrix} 2 & 5k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2(k+1) & (3k+4) \end{vmatrix} = 0$$

$$2\left((k+4)(3k+4) - (2k+2)(4k+2)\right) - 3k\left(3k+4 - (4k+2)\right)$$
$$+ 3k+4\left[2k+2 - (k+4)\right] = 0$$

$$2\left(3k^2+4k+12k+16 - (8k^2+4k+4k+2)\right] - 3k\left[3k+4-4k-2\right]$$
$$+ 3k+4\left(2k+2 - k - 4\right) = 0$$

$$2\left[3k^2+\cancel{4k}+12k+16 - 8k^2 - \cancel{4k} - 4k - 2\right] - 3k\left[-k+2\right]$$
$$+ 3k+4\left[k-2\right] = 0$$

$$2\left[-5k^2+8k+14\right] - 3k\left[-k+2\right] + 3k+4\left(k-2\right) = 0$$

$$-10k^2+16k+28+3k^2-6k+3k^2-6k+4k-12 = 0$$

$$-4k^2 + 4k + 16 = 0$$
$$4k^2 - 4k - 16 = 0$$

28 - 12
16

-10 + 3 + 3
-10 + 6
-4

16 - 6 - 6 + 4
16 - 6 - 9 + 4
-15

+16 - 6 - 6
+16 - 12

$$\frac{+5 \pm \sqrt{(-5)^2 - 4(4)(-16)}}{2(4)}$$

$$\frac{5 \pm \sqrt{25 + 386}}{8}$$

$$\frac{5 + \sqrt{411}}{8}$$

$$\frac{5 - \sqrt{411}}{8}$$

$$\Rightarrow \frac{5 \pm \sqrt{411}}{8}$$

$b^2 - 4ac = 400$

49 386

16

16 40 2

386

16

3 8 6

411

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Solve completely the following system of eq^n's

$$x + y + z = 1$$
$$x + 2y + 3z - 4 = 0$$
$$3y + x + 5z = 7$$
$$-10 + 4y + 7z + x = 0$$

$\rho(A) = \rho([A:B])$ = no. of variable ⇒ unique sol^n.
$\rho(A) = \rho([A:B])$ < no of variable = infinitely many
$\rho(A) \neq \rho([A:B])$ = no solution

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 5 \\ 1 & 4 & 7 \end{bmatrix} -4 \qquad B = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 1 & 5 \\ 1 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & -4 \\ 0 & 3 & 4 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

5-1

$$\sim \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 3 & 6 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_1 \\ R_4 \to 1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_4 \to R_4 - 3R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\rho(A) = 2$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 3 & : & 4 \\ 1 & 3 & 5 & : & 7 \\ 1 & 4 & 7 & : & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & 3 & : & 6 \\ 0 & 3 & 6 & : & 9 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & 2 & : & 5 \\ 0 & 3 & 6 & : & 9 \end{bmatrix} \sim \begin{bmatrix} \ \end{bmatrix}$$

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For what value of λ, the system of equation

$$x-2y-z=-1$$
$$3x-y+2z=1$$
$$y+\lambda z=1$$

has (1) no solution
(2) unique solⁿ
(3) Infinity many sol.

$$A=\begin{bmatrix}1 & -2 & -1\\ 3 & -1 & 2\\ 0 & 1 & \lambda\end{bmatrix},\ B=\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$A=\begin{bmatrix}1 & -2 & -1\\ 3 & -1 & 2\\ 0 & 1 & \lambda\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & -2 & -1\\ 0 & 5 & 5\\ 0 & 1 & \lambda\end{bmatrix} \quad R_2\to R_2-3R_1 \quad -1+6$$

$$\sim\begin{bmatrix}1 & -2 & -1\\ 0 & 5 & 5\\ 0 & 0 & \lambda-1\end{bmatrix} \quad R_3\to R_3-\frac{1}{5}R_2$$

$\lambda-1=0 \Rightarrow \lambda=1 \quad \rho(A)=2$

$$A:B=\begin{bmatrix}1 & -2 & -1 & : & -1\\ 3 & -1 & 2 & : & 1\\ 0 & 1 & \lambda & : & 1\end{bmatrix}$$

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$$\sim \left[\begin{array}{ccc:c} 1 & -2 & -1 & -1 \\ 0 & 5 & 5 & 4 \\ 0 & 1 & \lambda & 1 \end{array}\right] \quad R_2 \to R_2 - 3R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -2 & -1 & -1 \\ 0 & 5 & 5 & 4 \\ 0 & 0 & \lambda-1 & 1/5 \end{array}\right] \quad R_3 \to R_3 - \frac{1}{5}R_2$$

$$\rho(A:B) = 3$$

(1) $\rho(A) \neq \rho(A:B) = 3$

$\Rightarrow \lambda - 1 = 0, \ \lambda = 1$

(2) $\lambda - 1 \neq 0, \ \lambda \neq 1$

(3) There can never exist.

\#
$$x + y + 5z = 6$$
$$x + 2y + 3az = 5$$
$$x + 3y + az = 1$$

has (1) no solution
(2) unique solⁿ
(3) infinite "

sol:- $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3a \\ 1 & 3 & a \end{bmatrix}, \ B = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3a \\ 1 & 3 & a \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3a-5 \\ 0 & 2 & a-5 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$5 - a =$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3a-5 \\ 0 & 0 & -5a+5 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$\rho(A) = 3$

$$A:B = \begin{bmatrix} 1 & 1 & 5 & : & 6 \\ 1 & 2 & 3a & : & b \\ 1 & 3 & a & : & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & : & 6 \\ 0 & 1 & 3a-5 & : & b-6 \\ 0 & 2 & a-5 & : & -5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & : & 6 \\ 0 & 1 & 3a-5 & : & b-6 \\ 0 & 0 & -5a+5 & : & -2b+7 \end{bmatrix} \quad \begin{matrix} \cancel{R_3 \to R_3 - 2R_1} \\ R_3 \to R_3 - 2R_2 \end{matrix}$$

$\cancel{\rho(A) = 3}$ $\rho(A) = 3$

① $\rho(A) \neq \rho(A:B) = 3$ and $-2b+7 \neq 0$

and if $-5a+5 = 0$ $\qquad 2b = 7 \Rightarrow \boxed{b = 7/2}$

$a = -5/-5 = 1$ $\boxed{a = 1}$

② for unique soln, then

$-5a+5 \neq 0$ $\qquad -2b+7 = 0$

$a \neq 1$ $\qquad b = 7/2$

b any value

③ for infinite

$-5a+5 = 0$

$-2b+7 = 0$

$a = 1, b = 7/2$ Ans

For what value of λ and μ, the following system

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$ has

① $\lambda = 3, \mu \neq 10$

② $\lambda \neq 3$, μ any

③ $\lambda = 3, \mu = 10$

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Sol:- $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}$; $B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}$ $\lambda - 1 - 2$

$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & \lambda-1 \end{bmatrix}$ $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$\rho(A) = 3$

$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$ $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$\mu - 6 - 4$

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$ $R_3 \to R_3 - R_2$

$\rho(A) = 3$

① $\rho(A) \neq \rho(A:B) = 3$

and if : $\lambda - 3 \cancel{\neq} 0 \Rightarrow \lambda = 3$, $\mu - 10 = 0$, $\mu = 10$

$\boxed{\lambda = 3, \ \mu = 10}$

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(1) for unique solution
$\lambda - 3 \neq 0$, $\Rightarrow \lambda \neq 3$
$\mu - 10 \neq 0 \Rightarrow \mu \neq 10$

(3) infinite solⁿ
$\lambda - 3 = 0$, $\mu - 10 = 0$
$\lambda = 3$, $\mu = 10$ Ans

Applications of Linear Systems

Curve fitting
Given three points
(x_1, y_1)
(x_2, y_2)
(x_3, y_3)

To find the eqⁿ passing through given point.

Find the eqⁿ of polynomial curve which passes through points (1,4), (2,0), (3,12)

sol:-
$f(x) = a_0 + a_1 x + a_2 x^2$
$f(y) = a_0 + a_1 x + a_2 x^2$
$4 = a_0 + a_1 + a_2$
$0 = a_0 + 2a_1 + 4a_2$
$12 = a_0 + 3a_1 + 9a_2$

$AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \quad X = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$\rho(A) = 3$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 9 & 12 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 8 & 8 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 2 & 16 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

12 − 8

$\rho(A) = 3$

$\rho(A:B) = 3$

$\rho(A) = \rho(A|B)$ no. of variable.

so the system has unique sol.

~~$6a_2 = 4$~~ $2a_2 = 16 \to a_2 = 8$

~~$a_2 = 4 - 6 \to a_2 = -2$~~

$a_1 + 3a_2 = -4 \to a_1 + 3(8) = -4$

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$a_1 + 24 = -4$

$a_1 = -28$

$a_0 + a_1 + a_2 = 4$

$a_0 + (-28) + 8 = 4$

$a_0 - 28 + 8 = 4$

$a_0 = 4 + 20$

$a_0 = 24$

$\boxed{a_0 = 24,\ a_1 = -28,\ a_2 = 8}$

find eq of polynomial curve, which pass through point (0,0), (1,1) (3,3)

$f(x) = a_0 + a_1 x + a_2 x^2$

$0 = a_0$

$1 = a_0 + a_1 + a_2$

$3 = a_0 + 3a_1 + 9a_2$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ $\Rightarrow X = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -2 & 0 & 6 \end{bmatrix}$ $R_3 \to R_3 - 3R_2$

$-3 = -3$
$9 - 3$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 6 \end{bmatrix}$ $R_2 \to 6R_2 - R_3$

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$A:B \begin{bmatrix} 1 & 0 & 0 & : 0 \\ 1 & 1 & 1 & : 1 \\ 1 & 3 & 9 & : 3 \end{bmatrix} \sim \begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{0} & 2 & 8 \end{bmatrix}$ $\cancel{R_3 \to R_3 - R_1}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 0 & : 0 \\ 0 & 1 & 1 & : 1 \\ 0 & 3 & 9 & : 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & : 0 \\ 0 & 1 & 1 & : 1 \\ 0 & 1 & 3 & : 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & : 0 \\ 0 & 1 & 1 & : 1 \\ 0 & 0 & 2 & : 0 \end{bmatrix}$$

$2a_2 = 0$

$\boxed{a_2 = 0}$ $\boxed{a_1 = 1}$ $\boxed{a_0 = 0}$

dy

Find eq of polynomial curve which passes through point $(-2, 3)$, $(-1, 5)$, $(0, 1)$, $(1, 4)$, $(2, 10)$

$y = f(x)$

$y = f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$3 = a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4$

$5 = a_0 - a_1 + a_2 - a_3 + a_4$

$1 = a_0$

$4 = a_0 + a_1 + a_2 + a_3 + a_4$

$10 = a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4$

1+2 -1+2 1-4

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$$A = \begin{bmatrix} 1 & -2 & 4 & -8 & 16 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 4 \\ 10 \end{bmatrix} \quad X = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

A. $R_2 \to R_2 - R_1,\ R_4 \to R_4 - R_1,\ R_5 \to R_5 - R_1$

$$(A\ B) \begin{bmatrix} 1 & -2 & 4 & -8 & 16 & 3 \\ 0 & 1 & -3 & 7 & -15 & 2 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & -3 & 9 & -15 & 1 \\ 0 & 4 & 0 & 16 & 0 & 7 \end{bmatrix}$$

1- 1-16 2+2 8+8

$A'B$ $\left[\quad \right]$ $\cancel{R_2 \to R_2 + R_1}$ $-R$

$R_3 \leftrightarrow R_1$

$$A|B \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & : 1 \\ 0 & 1 & -3 & 7 & -15 & : 2 \\ 1 & -2 & 4 & -8 & 16 & : 3 \\ 0 & 3 & -3 & 9 & -15 & : 1 \\ 0 & 4 & 0 & 16 & 0 & : 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & : 1 \\ 0 & 1 & -3 & 7 & -15 & : 2 \\ 0 & -2 & 4 & -8 & 16 & : 3 \\ 0 & 3 & -3 & 9 & -15 & : 1 \\ 0 & 4 & 0 & 16 & 0 & : 7 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

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Find eqn of a polynomial cube which passes through points (1,1) (-4,-64), (2,8), (3,27)

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 7 & -15 & 2 \\ 0 & 0 & -2 & 6 & -14 & -37 \\ 0 & 0 & 6 & -12 & 30 & -5 \\ 0 & 0 & 12 & -12 & 60 & -1 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + 2R_2 \\ R_4 \to R_4 - 3R_2 \\ R_5 \to R_5 - 4R_2 \end{array}$$

47

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 7 & -15 & 2 \\ 0 & 0 & -2 & 6 & -14 & -37 \\ 0 & 0 & 0 & +6 & -12 & -16 \\ 0 & 0 & 0 & 12 & 0 & 9 \end{array}\right] \quad \begin{array}{l} R_5 + 6R_3 \\ R_5 \to R_5 - 2R_4 \\ R_4 \to R_4 + 3R_3 \end{array}$$

24 -24 41 $\rho(A:B) =$ 5

-12 +18

-30 +

$\rho(A) = \rho(A:B) =$ no. of variable

So the system has unique solutions

$12a_3 = 9$

$a_3 = 9/12 \Rightarrow a_3 = \frac{3}{4}$

$6a_2 - 12a_3 + a_4 30 = -5$

$6a_2 - 12(-3) + a_4 30 = -5$

$6a_2 + 36 + a_4 30 = -5$

$6a_2 + a_4 30 = -5 - 36$

$6a_2 + a_4 30 = -41$

$R_5 \to R_5 - 4R_4$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 7 & -15 & 2 \\ 0 & 0 & -2 & 6 & -14 & 7 \\ 0 & 0 & 0 & 6 & -12 & 16 \\ 0 & 0 & 0 & 0 & 18 & 19 \end{array}\right]$$

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$a_2 - \frac{1}{45} = 103 \Rightarrow a_2 = 103 + \frac{1}{45}$

$\frac{4635+1}{45} \Rightarrow \frac{4636}{45}$

103
9
513
412
4635

$$\left[\begin{array}{ccccc:c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 7 & -15 & 2 \\ 0 & 0 & -2 & 6 & -14 & 7 \\ 0 & 0 & 0 & 6 & -12 & 16 \\ 0 & 0 & 0 & 0 & 18 & 19 \end{array}\right]$$

$18a_4 = 19 \Rightarrow a_4 = \frac{19}{18}$

$6a_3 - 12a_4 = 16 \Rightarrow 6a_3 - 12\left(\frac{19}{18}\right) = 16$

$6a_3 - \frac{57}{2} = 16 \Rightarrow 6a_3 = 16 + \frac{57}{2} \Rightarrow 6a_3 = \frac{32+57}{2}$

57
+32
=89

$a_3 = \frac{32+57}{2\times 6} \Rightarrow \frac{89}{12} \Rightarrow$

~~Equation of a polynomial curve which passes through points~~

$a_3 = \frac{89}{12}$

$-2a_2 + 6a_3 - a_4 14 = 7$

81

$-2a_2 + 6\left(\frac{89}{12}\right) - \left(\frac{19}{18}\right)14 = 7$

$\frac{19}{2}$
151

$-2a_2 + \frac{89}{2} - \frac{151}{9} = 7$

89
9

$-2a_2 + \frac{801 - 302}{18} = 7$

$-2a_2 + \frac{499}{18} = 7 \Rightarrow -2a_2 = 7 - \frac{499}{18}$

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$$-2a_2 = \frac{126 - 499}{18}$$

$$\cancel{-}2a_2 = \frac{\cancel{-}373}{18}$$

$$\cancel{-}a_2 = \frac{373}{18 \times 2} \Rightarrow a_2 = \frac{373}{36}$$

$$a_1 - 3a_2 + 7a_3 - 15a_4 = 2$$

$$a_1 - \cancel{3}\left(\frac{373}{\cancel{36}\ 12}\right) + 7\left(\frac{89}{12}\right) - 15\left(\frac{19}{18}\right) = 2$$

$$a_1 - \frac{373}{12} + \frac{623}{12} - \frac{285}{18} = 2$$

$$a_1 = 2 + \frac{373}{12} - \frac{623}{12} + \frac{285}{18}$$

$$a_1 = \frac{72 + 1119 - 1869 + 570}{36}$$

$$a_1 = \frac{-108}{36}$$

$$a_1 = -3 \qquad , a_0 = 1$$

$$f(x) = \cancel{a_0}\,1 + (-3)x + \frac{373}{36}x^2 + \frac{89}{12}x^3 + \frac{19}{18}x^4$$

$$f(x) = 1 - 3x + \frac{373}{36}x^2 + \frac{89}{12}x^3 + \frac{19}{18}x^4$$

$a_0 + a_1x + a_2x^2 + a_3x^3$

-4×4
16×4

$(1,1), (-4,-64), (2,8), (3,27)$

$$y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$
$$1 = a_0 + a_1 + a_2 + a_3$$
$$-64 = a_0 - 4a_1 + 16a_2 - 64a_3$$
$$8 = a_0 + 2a_1 + 4a_2 + 8a_3$$
$$27 = a_0 + 3a_1 + 9a_2 + 27a_3$$

$$(A:B) \begin{bmatrix} 1 & 1 & 1 & 1 & : & 1 \\ 1 & -4 & 16 & -64 & : & -64 \\ 1 & 2 & 4 & 8 & : & 8 \\ 1 & 3 & 9 & 27 & : & 27 \end{bmatrix}$$

$-64-1$

$$(A:B) \begin{bmatrix} 1 & 1 & 1 & 1 & : & 1 \\ 0 & -5 & 15 & -65 & : & -65 \\ 0 & 1 & 3 & 7 & : & 7 \\ 0 & 2 & 8 & 26 & : & 26 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$R_3 \to 5R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 1 \\ 0 & -5 & 15 & -65 & : & -65 \\ 0 & 0 & 30 & 30 & : & 30 \\ 0 & 2 & 8 & 26 & : & 26 \end{bmatrix} \quad \begin{array}{l} \cancel{R_3 \to R_3 +} \\ \cancel{R_4 \to R_4 - 2R_1} \\ \cancel{R_3} \to \end{array}$$

$35 - 65$
30

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 1 \\ 0 & -5 & 15 & -65 & : & -65 \\ 0 & 0 & 30 & 30 & : & 30 \\ 0 & 1 & 4 & 13 & : & 13 \end{bmatrix}$$

$\frac{13}{65}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 1 \\ 0 & -5 & 15 & -65 & : & -65 \\ 0 & 0 & 30 & 30 & : & 30 \\ 0 & 0 & 45 & 0 & : & 0 \end{bmatrix} \quad R_5 \to 5R_5 + R_2$$

$20 + 15$
$65 - 60$

$\frac{13}{65}$

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$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & -5 & 15 & -65 & -65 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 45 & 0 & 0 \end{array}\right]$$

$|A| : A : B)$

$a_2 = 0$

$a_2 + a_3 = 1$

$0 + a_3 = 1 \quad a_3 = 1$

$-5a_1 + 15a_2 - 65a_3 = -65$

$-5a_1 + 15(0) - 65a_3 = -65$

$-5a_1 - 65a_3 = -65$

$\cancel{5a_1 - 15a_2}$

$-5a_1 - 65 = -65$

$-5a_1 = -65 + 65$

$-5a_1 = 0$

$a_1 = 0$

$5 | 65 \; 13$

$a_0 + a_1 + a_2 + a_3 = 1$

$a_0 + 0 + 0 + 1 = 1$

$a_0 = 1 - 1$

$a_0 = 0$

$y = f(x) = x^3$

$\boxed{y = x^3}$

$(1, 1), (-4, -64), (2, -8), (3, 27)$

$y = f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$1 = a_0 + a_1 + a_2 + a_3$

$-64 = a_0 - 4a_1 + 16a_2 - 64a_3$

$-8 = a_0 + 2a_1 + 4a_2 + 8a_3$

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$27 = a_0 + 3a_1 + 9a_2 + 27a_3$

$$(A:B) \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 1 & -4 & 16 & -64 & -64 \\ 1 & 2 & 4 & 8 & -8 \\ 1 & 3 & 9 & 27 & 27 \end{array}\right]$$

$$(A:B) \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & -5 & 15 & -65 & -65 \\ 0 & 1 & 3 & 7 & -9 \\ 0 & 2 & 8 & 26 & 26 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$-3 - 1$

$$A:B \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & -5 & 15 & -65 & -65 \\ 0 & 0 & 30 & 30 & -105 \\ 0 & 2 & 8 & 26 & -26 \end{array}\right] \quad R_3 \to 5R_3 + R_2$$

$(-9)(5) + (-65)$

$-45 - 65$

65
$\underline{40}$
105

$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & -5 & 15 & -65 & -65 \\ 0 & 0 & 30 & 30 & -105 \\ 0 & 1 & 4 & 13 & 13 \end{array}\right]$$

30

$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & -5 & 15 & -65 & -65 \\ 0 & 0 & 30 & 30 & -105 \\ 0 & 0 & 45 & 0 & 0 \end{array}\right] \quad R_5 \to 5R_5 + R_2$$

$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & -5 & 15 & -65 & -65 \\ 0 & 0 & & & \\ & & & & \end{array}\right]$$

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$\boxed{a_2 = 0}$

$30a_2 - 30a_3 = -105$

$\cancel{-}30a_3 = \cancel{-}105$

$30a_3 = 105$

$a_3 = \frac{\cancel{105}}{\cancel{30}} \frac{21}{6} = \frac{21}{6}$

$\boxed{a_3 = \frac{21}{6}}$

$-5a_1 + 15a_2 - 65a_3 = -65$

$-5a_1 - 65\left(\frac{21}{6}\right) = -65$

$-5a_1 - \frac{\cancel{1365}^{455}}{\cancel{6}_{2}} = -65$

$-5a_1 - 455 = -65$

$-5a_1 = -65 + 455$

$-5a_1 = 390$

$-a_1 = \frac{390}{5}$

$\boxed{a_1 = -78}$

$a_0 + a_1 + a_2 + a_3 = 1$

$a_0 - 78 + \frac{21}{6} = 1 \Rightarrow a_0 = 1 - \frac{21}{6} + 78$

$a_0 = \frac{6 - 21 + 468}{6}$

474
453
78 × 6
468 + 6
474

$a_0 = \frac{474 - 21}{6}$

$a_0 = \frac{453}{6}$

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Eigen Values

Let A be a square matrix of order n. A scalar λ is called an eigen value of A if

$$AX = \lambda X \text{ for some non-zero vector}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

characteristic value,
Spectral value
latent value

$$\lambda X = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}$$

Working method to find eigen value

1. Let $A_{n \times n}$ be the given matrix
2. Consider $I_{n \times n}$
3. Put $|A - \lambda I| = 0$

example $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

E. $(A - \lambda I) = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix}$

$(1-\lambda)(3-\lambda) = 0 \quad \Rightarrow \lambda = 1, 3$

→ we get eq^n of degree n. It is called characteristic equation of matrix A

4. Find root of characteristic equation

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If $A = \begin{bmatrix} a & & & & \\ & b & & 0 & \\ & & c & & \\ & 0 & & d & \\ & & & & e \end{bmatrix}$

$[A - \lambda I] = 0 \quad \begin{vmatrix} a-\lambda & & & & \\ & b-\lambda & & & \\ & & c-\lambda & 0 & \\ & & & d-\lambda & \\ & 0 & & & e-\lambda \end{vmatrix}$

$(a-\lambda)(b-\lambda)(c-\lambda)(d-\lambda)(e-\lambda) = 0$

$\lambda = a, b, c, d, e$

∴ The eign value are the diagonal value itself.

$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix} \Rightarrow I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$[A - \lambda I] = 0$

$\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{bmatrix} = 0$

$[(2-\lambda)((-3-\lambda)(1-\lambda)) - 2] - 2(2(-3-\lambda) + 7) = 0$

$[(2-\lambda)((-3+3\lambda-\lambda+\lambda^2) - 2] - 2(-6-2\lambda+7)$

$(2-\lambda)[-5+2\lambda+\lambda^2) - 2(\cancel{7}1-2\lambda) = 0$

$-10 + 4\lambda + \cancel{2\lambda^2} + 5\lambda - \cancel{2\lambda^2} - \lambda^3 - 2 + 4\lambda = 0$

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$-\lambda^3+13\lambda-12=0$

$\lambda^3-13\lambda+12=0$

$$\lambda-1 \overline{\big) \lambda^3-13\lambda+12} \big(\lambda^2-\lambda-12$$

$$\cancel{\lambda^3}-\lambda^2$$

$$-13\lambda+12+\lambda^2$$

$$-\lambda \quad -\lambda^2$$

$$-12\lambda+12$$

$$-12\lambda-12$$

$(\lambda-1)(\lambda^2+\lambda-12)=0$

$(\lambda-1)(\lambda^2+4\lambda-3\lambda-12)=0$

$(\lambda-1)\,\lambda(\lambda-4)-3(\lambda+4)=0$

$\lambda=1,3,-4$

Spectrum of a matrix

collection of all eign values of matrix is called its spectrum.

spectrum of A = $\{1,3,-4\}$
above eq

\# $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & k \end{bmatrix}$ are $0, 3, 15$ find the value of k.

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if λ is eign value of A, then

$|A-\lambda I| = 0$ $\qquad |A-3I| = 0$

$|A-0I| = 0$ $\qquad |A-15I| = 0$

$|A| = 0$

$|A-0I| = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & k \end{bmatrix} = 0$$

$$8[7k-16) + 6[-6k+8] + 2[24-14] = 0$$

$$56k - 128 - 36k + 48 + 48 - 28 = 0$$

$$20k - 60 = 0$$

$$20k = 60$$

$$k = \cancel{60}\,3 / \cancel{20}$$

$$\boxed{k = 3}$$

Find the spectrum and characteristic e^n of following matrix

(1) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(2) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

(5) $\begin{bmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

(1)

$$A \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$2-\lambda[(2-\lambda)(2-\lambda) - 1] + 1[(-1)(2-\lambda) + 1] + 1[1 - (2-\lambda)] = 0$$

$$2-\lambda[4-2\lambda-2\lambda-\lambda^2-1] + 1[-2+\lambda+1] + 1[1-2+\lambda] = 0$$

$$2-\lambda[3-4\lambda-\lambda^2] + 1[-1+\lambda] + 1[-1+\lambda] = 0$$

$$6 - 8\lambda - 2\lambda^2 - 3\lambda + 4\lambda^2 + \lambda^3 - 1 + \lambda - 1 + \lambda = 0$$

$$\lambda^3 + 2\lambda^2 - 9\lambda + 4 = 0$$

λ^2

$-8-3+1+1$

$-11+2$

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Singular matrix

Exercise (1)

Show that atleast one eigen value of ~~every singular~~ a matrix is 0, iff the matrix is singular.

Proof:

$$|A| = 0$$

$$|A - 0I| = 0$$

0 is eigen value

Conversely let 0 be eigen value of A

$$|A - 0I| = 0$$

$$|A| = 0$$

A is singular

(2) A and A^T have same eigen value

Proof: Eigen value of A^T are sol. of

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$|A^T - \lambda I| = 0$

$|A^T - (\lambda I)^T| = 0$

$|(A - \lambda I)^T| = 0$

$\Rightarrow |A - \lambda I| = 0$

$\therefore$ eigen values of A^T are solutions of characteristic eq^n of A

$\therefore$ eigen values of A^T and A are same

(iii) If λ is eigen value of non-singular matrix A then $\frac{1}{\lambda}$ is eigen value of A^{-1}

Proof: Let λ be eigen value of A
So there exist non-zero X such that
$$AX = \lambda X$$
Now A^{-1} exist and so $\lambda \neq 0$
$$\frac{1}{\lambda} AX = X$$
$$A\left[\frac{1}{\lambda} X\right] = X$$
$$\frac{1}{\lambda} X = A^{-1} X$$
$\Rightarrow \frac{1}{\lambda}$ is eigen value of A^{-1}

Q Find eigen values and corresponding eigen vector of matrix
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2\times 2}$$

$|A - \lambda I| = 0$
$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

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$(1-\lambda)(4-\lambda) - 4 = 0$

$4 - \lambda - 4\lambda + \lambda^2 - 4 = 0$

$-5\lambda + \lambda^2 = 0$

$\lambda^2 - 5\lambda = 0$

$\lambda[\lambda - 5] = 0$

$\lambda = 0, 5$

To find eigen vectors :-

$$\underset{\text{eigen vector}}{A\overset{}{x}} = \underset{\text{eigenvalue}}{\lambda} x$$

eigen vector corresponding to $\lambda = 0$

$Ax = 0x$

$Ax = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

Put $x_2 = 1$

eigen vector $= \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

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eigen vector corresponding to $\lambda = 5$

$$AX = 5X$$

$$(A - 5I)\lambda = 0$$

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

$$R_2 \to 2R_2 + R_1$$

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}$$

$$-4x_1 + 2x_2 = 0$$

$$2[-2x_1 + x_2] = 0$$

$$-2x_1 + x_2 = 0$$

$$-2x_1 = -x_2$$

$$2x_1 = x_2 \qquad x_1 = \frac{1}{2}x_2$$

$$\cancel{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\begin{bmatrix} x_2 \\ x_2 \end{bmatrix}} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \quad \end{bmatrix} x_2$$

Put $x_2 = 1$

Q. Find eigen values and corresponding eigen value of matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} = 0$$

$(1-\lambda)[(2-\lambda)(3-\lambda)-2] - 1[2-2(2-\lambda)]$

$(1-\lambda)[6-2\lambda-3\lambda+\lambda^2-2] - 1(2-4+2\lambda)$

$(1-\lambda)(4-5\lambda+\lambda^2) - 1(-2+2\lambda)$

$(4-5\lambda+\lambda^2-\lambda 4+5\lambda^2-\lambda^3) + 2 - 2\lambda = 0$

$4 - 9\lambda + 6\lambda^2 - \lambda^3 + 2 - 2\lambda = 0$

$6 - 11\lambda + 6\lambda^2 - \lambda^3 = 0$

$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 2 = 0$

$\lambda - 1 \overline{)\, \lambda^3 - 6\lambda^2 + 8\lambda - 2\,}\ (\lambda^2 - 5\lambda + 3$

$-8\lambda \quad \lambda^3 - \lambda^2$

$- \quad +$

$-5\lambda^2 + 8\lambda - 2$

$-8\lambda^2 \pm 5\lambda$

$+ \quad -$

$3\lambda - 2$

$3\lambda - 3 \qquad -5$

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$(\lambda-1)(\lambda^2-5\lambda+6)=0$

$(\lambda-1)[\lambda^2-2\lambda-3\lambda+6]=0$

$(\lambda-1)[\lambda(\lambda-2)-3[\lambda-2]]=0$

$(\lambda-1)(\lambda-2)(\lambda-3)=0$

$\lambda=1,2,3$

To find eigen vector

$AX=\lambda X$

eigen vector corresponding to $\lambda=1$

$AX=X \Rightarrow (A-I)X=0$

$$\begin{bmatrix}1&0&-1\\1&2&1\\2&2&3\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}X=0$$

$$\begin{bmatrix}0&0&-1\\1&1&1\\2&2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$A=\begin{bmatrix}0&0&-1\\1&1&1\\2&2&2\end{bmatrix}$$

$R_3\to R_3-2R_2$

$$A=\begin{bmatrix}0&0&-1\\1&1&1\\0&0&0\end{bmatrix}$$

$-x_3=0$ — (1) $\Rightarrow x_3=0$ — Put in (2)

$x_1+x_2+x_3=0$ — (2)

$x_1+x_2=0$

$x_1=-x_2$ or $-x_1=x_2$

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$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} x_1$$

Put $x_1 = 1$ $\quad \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

eigen vector corresponding to $\lambda = 2$

$AX = 2X \Rightarrow (A - 2I)X = 0$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} X = 0$$

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} X = 0$$

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

$-x_1 - x_3 = 0 \quad \Rightarrow -x_1 = x_3 \quad$ — (1) Put in (2)

$2x_2 - x_3 = 0 \quad \Rightarrow 2x_2 = x_3 \quad$ — (2)

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$2x_2 = -x_1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 2x_2 \end{bmatrix}$$

$$\begin{bmatrix} -2x_2 \\ x_2 \\ 2x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} x_2 = 0 \Rightarrow x_2 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

eigen vector corresponding to $\lambda = 3$

$Ax = 3x \Rightarrow (A - 3I)x = 0$

$$\left(\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) x = 0$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} x = 0$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x = 0$$

$R_2 \to 2R_2 + R_1$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix} x = 0$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$-2x_1 - x_3 = 0$

$-2x_2 + x_3 = 0$

$-x_3 = 0$

$\Rightarrow -2x_1 = x_3$

$-2x_2 = x_3$

$x_3 = 0$

$-1+3$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 3 \\ 0 & 0 & 2 \end{bmatrix} x = 0$$

$-2x_1 - x_3 = 0 \Rightarrow -2x_1 = x_3$

$-2x_2 + 3x_3 = 0 \Rightarrow -2x_2 = -3x_3$

$2x_3 = 0 \qquad 2x_3 = 0$

$-2x_1 - x_3 = 0$

$-2x_2 + x_3 = 0$

$-2x_1 = x_3 \qquad x_1 = -\frac{1}{2}x_3$

$-2x_2 = -x_3 \Rightarrow 2x_2 = x_3$

$x_2 = \frac{1}{2}x_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} \\ +\frac{1}{2} \\ 1 \end{bmatrix} x_3 = 0$$

$x_3 = 1$

$$\begin{bmatrix} -1 \\ +1 \\ 2 \end{bmatrix} = 0$$

Find eigen values characteristic equation and eigen ~~val~~ vector of the matrix.

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$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$(A - \lambda I) = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$6-\lambda\left[(3-\lambda)(3-\lambda)-1\right] + 2\left[(-2)(3-\lambda)+2\right] + 2\left[2-(2)(3-\lambda)\right] = 0$$

$$6-\lambda\left[9-3\lambda-3\lambda+\lambda^2-1\right] + 2\left[-6+2\lambda+2\right] + 2\left[2-6+2\lambda\right] = 0$$

$$54-18\lambda-18\lambda+6\lambda^2-6-9\lambda+3\lambda^2+3\lambda^2-\lambda^3+\lambda-12+4\lambda+4$$
$$+4-12+4\lambda = 0$$

$$\cancel{-\lambda^3+42\lambda^2-28\lambda+16} = 0 \qquad \lambda^3-12\lambda^2+36\lambda-32 = 0$$

$$\cancel{\lambda^3-42\lambda^2+28\lambda-16=0}$$

$$\lambda-2 \,\big)\, \lambda^3-12\lambda^2+36\lambda-32 \,\big(\, \lambda^2-10\lambda+16$$
$$\lambda^3-2\lambda^2$$
$$-10\lambda^2+36\lambda-32$$
$$-10\lambda^2+20\lambda$$
$$16\lambda-32$$
$$16\lambda-32$$
$$\times$$

$$(\lambda-2)(\lambda^2-10\lambda+16)$$
$$(\lambda-2)\left[\lambda^2-8\lambda-2\lambda+16\right]$$
$$(\lambda-2)\left[\lambda[\lambda-8]-2[\lambda-8]\right]$$

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eigen value 2, 2, 8

$\lambda = 2$ eigen value of corresponding to $\lambda = 2$

$Ax = 2X$

$(A - 2I)X = 0$

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} X = 0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} X = 0$$

$R_2 \to 2R_2 + R_1$, $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$4x_1 - 2x_2 + 2x_3 = 0$

$2x_1 - x_2 + x_3 = 0$

$2x_1 - x_2 = -x_3$

$-x_2 = -x_3 - 2x_1$

$-x_2 = -[x_3 + 2x_1]$

$x_2 = x_3 + 2x_1$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ x_3 + 2x_1 \\ x_3 + 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ 2x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} x_3 = 0$$

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Put x_1 and $x_3 = 1$

so $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are L.I eigen vector corresponding to $\lambda = 2$

eigen value of corresponding to $\lambda = 8$

$$AX = 8X$$

$$(A - 8I)X = 0$$

$$\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \right) X = 0$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} X = 0$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_2$

$-5+2$ $\quad -1-2 \quad -1-5 \quad -5-$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} = 0$$

$R_3 \to R_3 - 2R_2$

$-6+$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} X = 0$$

$-2x_1 - 2x_2 + 2x_3 = 0 \quad$ (1)

$-3x_2 - 3x_3 = 0 \quad$ (2)

from (1)

$-x_1 - x_2 + x_3 = 0 \Rightarrow \cancel{-x_1 = -x_3 + x_2}$

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$\cancel{-x_1 = -(x_3 - x_2)}$ $\quad$ $\cancel{-x_2 = -x_3 + x_1}$

$\boxed{\cancel{x_1 = x_3 - x_2}}$ $\quad$ $\boxed{x_3 = x_3 - x_1}$

$\Rightarrow \cancel{x_1 + x_3 = x_2}$

from (2) $\quad$ $x_2 = -x_2 - x_1$

$-3x_2 - 3x_3 = 0$

$x_2 + x_3 = 0$

$x_2 = -x_3 \Rightarrow -x_2 = x_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 - x_1 \begin{bmatrix} x_3 - x_2 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} - \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -7 & 5 & 1 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 3 & 1-\lambda & 0 \\ -7 & 5 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[(1-\lambda)(1-\lambda)] = 0$

$(1-\lambda)(1-\lambda-\lambda+\lambda^2) = 0$

$1-\lambda-\lambda+\lambda^2-\lambda+\lambda^2+\lambda^2-\lambda^3 = 0$

$1-3\lambda+3\lambda^2-\lambda^3 = 0$

$-\lambda^3+3\lambda^2-3\lambda+1 = 0$

$\lambda^3-3\lambda^2+3\lambda-1 = 0$

$\lambda - 1 \overline{)\ \lambda^3-3\lambda^2+3\lambda-1\ }\ (\lambda^2-2\lambda+1$

$-\lambda$

$-2\lambda^2+2\lambda$

$\lambda^3 - \lambda^2$

$-2\lambda^2+3\lambda-1$

$-2\lambda^2+2\lambda$

$\lambda - 1$

$\lambda - 1$

0

$(\lambda-1)(\lambda^2-2\lambda+1) = 0$

$(\lambda-1)(\lambda^2-\lambda-\lambda+1)$

$(\lambda-1)\lambda(\lambda-1)-(\lambda-1)$

$(\lambda-1)(\lambda-1)(\lambda-1)$

$\lambda = 1, 1, 1$

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$\lambda = 1, 1, 1$

eigen value of corresponding to $\lambda = 1$

$AX = X$

$(A - I)x = 0$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -7 & 5 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) x = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ -7 & 5 & 0 \end{bmatrix} x = 0$$

$3x_1 = 0 \Rightarrow x_1 = 0$

$-7x_1 + 5x_2 = 0 \Rightarrow 5x_2 = 0$

$x_2 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad x = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3$$

Ans

Unit - 4 Relation

(1) Relation

from A to B

Let A and B be two sets. A relation R is a subset of $A \times B$.

$A = \{1, 2, 3\}$

$B = \{x, y\}$

$A \times B = \{(1,x), (1,y), (2,x), (2,y), (3,x), (3,y)\}$

$B \times A = \{(x,1), (y,1), \ldots\}$

$S \subseteq B \times A$

↓

Relation from B to A

$|A|$ = no. of element in A

$\Rightarrow (A \times B) = |A| \times |B|$

$= 3 \times 2 = 6$

Subset $|A \times B|$

$2 = 2$

$A \times B = S$ $\quad = B \times A = T$

- Let A ~~is~~ No. of relations from A to B = No. of subsets of $A \times B$

= No. of subsets of S

$= 2^{|S|}$

$= 2^{|A \times B|} = 2^{|A| \times |B|}$

- No. of relations from B to A = No. of subset, $B \times A$

= No. of subset of T

$= 2^{|T|} = 2^{|B \times A|}$

$= 2^{|B| \times |A|}$

$= 2^{|A| \times |B|}$

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$A = \{1,2,3,4\}$, $B = \{x, y\}$.

$A \times B = \{(1,x)\ (1,y), (2,x), (2,y)(3,x), (3,y), (4,x)$
$(4,y)\}$

$\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}$
— — — — — .

$A = \{1\}$, $B = \{x, y\}$
$A \times B = \{(1,x)\ (1,y)\}$.

$S = \phi, \{(1,x)\}, \{(1,y)\}, \{(1,x), (1,y)\}\}$

(i) Reflexive Relation :- Let A be a set A relation from A to A is called reflexive if $(x,x) \in R\ \forall\ x \in A$

$A = \{1,2,3\}$
$A \times A = \{(1,1)\ (1,2), (1,3)\ (2,1), (2,2), (2,3), (3,1)(3,2),$
$(3,3)\}$.

$S = (1,1), (2,2), (3,3)$

A relation on a set A is said to be reflexive if $aRa\ \forall\ a \in A$

eg:- $A = \{x, y, z\}$
$S = (x,x), (x,y), (y,x), (y,y)$.
$S \subseteq A \times A$

sol:-

(ii) Symmetric Relation :-

A relation defined on set A is said to be symmetric if whenever aRb, then

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bRa $\forall$ a, b $\in$ A

(iii) Transitive Relation :- A relation defined on a set A is said to be transitive if aRb & bRc $\Rightarrow$ aRc

$A = \{x, y, z\}$

$S = \{(x,x), (y,y) (z,z)\}$
$S = \{(x,x), (y,y), z(z), (x,y) (y,z)\}$

(iv) Anti-Symmetric relation :- A relation defined on a set A is said to be anti-symmetric if aRb and bRa $\Rightarrow$ a = b

$A = \{1, 2, 3\}$
$S = \{(1,1), (1,2), (2,1)\}$ (1) (2,2)

Partial Order Relation

A relation on set A is said to be partial order relation if it is

(1) Reflexive aRa
(2) Anti-Symmetric aRb bRa $\Rightarrow$ a = b
(3) Transitive

Equivalence relation :-

(1) Reflexive
(2) Symmetric
(3) Transitive.

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Example I. Let R be relation defined on set $X = \{0, 1, 2, 3, \ldots\}$ of non-negative integers defined by equation $x^2 + y^2 = 25$. Write R as set of ordered pairs.

Sol:- $R = \{(x, y), x, y \in X, x^2 + y^2 = 25\}$

$\{(0,5), (3,4), (4,3), (5,0)\}$

II. Let R be relation on set N of positive integers defined by equation $3x + 4y = 17$. Write R as a set of ordered pairs.

Sol:- $R\ \{(x, y), x, y \in N, 3x + 4y = 17\}$

$R = \{(3,2)$

III Let R be a relation on the set of all lines of a plane, defined by $(l_1, l_2) \in R$ iff $l_1 \parallel l_2$. Is R equivalence?

(1) Reflexive

lRl $\forall$ l is plane

(2) Symmetric

$l_1 R l_2 \Rightarrow l_1 \parallel l_2 \Rightarrow l_2 \parallel l_2$

$= l_2 R l_1$

(3) Transitive

Let $l_1 R l_2$ and $l_2 R l_3$

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$l_1 \parallel l_2$ and $l_2 \parallel l_3$

$l_1 \parallel l_3 \Rightarrow l_1 R l_3$

Show that the relation "is congruent to" on the set of all triangles is an equivalence relation?

(1) Relation ~~Reflexive~~ :- Transitive

Let $T_1 R T_2$ and $T_2 R T_3$

$T_1 \cong T_2$ and $T_2 \cong T_3$

So $T_1 \cong T_3$

Let A be the set of integers and relation $\sim$ is defined on $A \times A$ by $(a,b) \sim (c,d)$ iff $a+d = b+c$. Prove that this is ~~equi~~ equivalence

Sol:- Reflexive :- if $(a,b) R (a,b)$

$(a,b) R (a,b)$ iff $a+b = b+a$, which is true

$\Rightarrow (a,b) R (a,b)$

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Symmetric: T.P If $(a,b) R (c,d)$ then $(c,d) R (a,b)$

Assume that $(a,b) R (c,d)$

$\Rightarrow a+d = b+c$

$\Rightarrow d+a = c+b$

$= c+b = d+a$

$= (c,d) R (a,b)$

Transitive:- T.P If $(a,b) R (c,d)$ and $(c,d) R (e,f)$

then, $(a,b) R (e,f)$

Assume that $(a,b) R (c,d)$ and $(c,d) R (e,f)$

$\Rightarrow a+d = b+c$ and $c+f = d+e$

add both

$a+\cancel{d}+\cancel{c}+f = b+\cancel{c}+\cancel{d}+e$

$a+f = b+e$

$(a,b) R (f,e)$

\# construct an example of relation on a set A which is reflexive but not symmetric.

Sol:- $A = \{1,2,3\}$

~~$A \times A$ {(1,1), (1,2) (1,3), (2,1) (2,2) (2,3) (3,3) (3,2) (3,1)}~~

$= \{(1,1), (2,2), (3,3), 1,2\}$

It is reflexive but not a symmetric

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construct an example of relation on a set A which is ~~reflexive~~ symmetric but not ~~symmetric~~ transitive

Sol:- $A = \{1,2,3\}$

$A \times A = \{\cancel{(1,1)}, \cancel{(1,2)}, (1,3), \cancel{(2,1)}, \cancel{(2,2)}, (2,3), \cancel{(3,1)}, (3,2), (3,3)\}$

$= \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$

It is reflexive or symmetric but not a transitive.

Symm and ~~reflexive~~ transitive but not reflexive

$A = \{1,2,3\}$

$A \times A = \{(2,1), (2,2), (1,2)\}$.

Symm but not transitive ~~but~~ nor reflexive

$A = \{1,2,3\}$

$S = \{(1,1), (2,2)\}$

Function

every function is rel but every relation is not a function

Let A and B be two sets
Then function from $A \to B$ is a subset of $A \times B$ st such that

(i) $\forall a \in A, (a, b) \in f$ if some $b \in B$
(ii) if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$

q. $A = \{1, 2, 3\}$
$B = \{x, y, z\}$
$f \subseteq A \times B$
$f = \{(1, x), (2, x), (3, y)\}$

$f : A \to B$
& B $f(A) \to$ range
A = domain
B = co-domain

Give an example of a relation which is not function.

$A = \{1, 2, 3\}$
$\{(1, 1), (1, 2)\}$

~~Give an example~~ : (competitive)
(1) One-One function :- function $f : A \to B$ is called oneone if

whenever $f(a_1) = f(a_2)$
then $a_1 = a_2$

or if $(a_1, b_1) = (a_2, b_1)$ then $a_1 = a_2$

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$-(\cancel{\text{Surpective}}$ Surjective

② **onto function** :- A function $f: A \to B$ is called onto if for given $b \in B$, $\exists\ a \in A$ such that

$$b = f(a)$$

or $\forall b \in B, (a, b) \in f$ for some $a \in A$

\# Let $A = \{1, 2, 3\}$, $B = \{3, 6, 11, 9\}$ and $f: A \to B$ be defined by $f(x) = x^2 + 2$
$g: A \to B$ is going by $g(x) = 3x$
Is $f(x) = g(x)$ for some $x \in A$?

Sol:- $f(1) = 1^2 + 2 = 3$
$g(1) = 3$
$f(2) = 4 + 2 = 6$
$g(2) = 3(2) = 6$

∴ yes, $1 \in A$ such that
$f(1) = g(1)$
and $2 \in A$ such that
$f(2) = g(2)$
$f = g$ iff $f(x) = g(x)\ \forall x \in$ Domain

for $x = 3$, $f \neq g$ ∵ $f(3) \neq g(3)$

\# which of the following function are one one or onto or both?
(1) $f: R \to R$ defined by $f(x) = x^3$
(2) $f: Z \to Z$ defined by $f(x) = 2 - x$
(3) $f: N \to N$ defined by $f(m, n) = 2^m 3^n$

Sol:- ① Let $f(x) = x^3$
$f(x_1) = f(x_2)$
$x_1^3 = x_2^3$

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$(x_1^3 - x_2^3) = 0$

$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$x_1^2 + x_1x_2 + x_2^2 = 0$

So that function is one-one

(ii) Let $y \in R$

T.P $\exists x \in R$ such that

$f(x) = y$

$x^3 = y$

$x = y^{1/3} \in R$

So that the function is onto.

(11) (i) $f(x) = 2 - x$

$x_1 = x_2 \Rightarrow 2x_1 - x_1 = 2x_2 - x_2$

~~$x_1 = x_2$~~

$\cancel{+}x_1 = \cancel{+}x_2$

$x_1 = x_2$

$\therefore$ one-one function

(ii) Let $y \in Z$

T.P $\exists x \in Z$ such that

$f(x) = y$

$2 - x = y$

$-x = y - 2$

$x = 2 - y \in Z$

$\therefore$ onto function

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(ii) (i) $f(m_1) = f(m_2)$

$\Rightarrow 2^{m_1} = 2^{m_2}$

$\Rightarrow 2^{m_1 - m_2} = 1$ $\quad [2^0 = 1]$

$m_1 - m_2 = 0$

$m_1 = m_2$

So it is one-one function

(ii) Onto

$1 \in N$

$f(x) = y$

If f is onto, then $\exists$ a natural no. n such that

$f(n) = 1$

$2^n - 1 = 2^0$

$n = 0$

but $0 \notin N$

Let $A = B = \{1, 2, 3, 4, 5\}$

Define $f : A \rightarrow B$ such that

(1) f is one-one but not onto.

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cardinality of a sol^n:-

Definition:- Let A be a set then the cardinality of A is number of elements in A or (order of set A)

Two finite sets are said to have same cardinality if they have same no. of elements

$A = \{1, 2, 3\}$

$B = \{x, y\}$

~~Two infinite sets are said to have same cardinality if they have same no. of elements~~

Two infinite sets are said to have same cardinality if there is a one-one onto mapping from one set to another.

One- one corresponding in A & B

One-one onto mapping from A to B

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 5\}$

$A = \mathbb{N}, \quad B = W$

$f(n) = n - 1$

well defined : If $x_1 = x_2$

$f(x_1) = f(x_2)$

one one: $f(x_1) = f(x_2)$

$x_1 = x_2$

onto:- $y \in Y$

then $\exists\ x \in X$ such that

$y = f(x)$

Z = Integers
$2Z$ = even integers

Q. well-defined : $f(n) = n-1$

let $n_1 = n_2$

$n_1 - 1 = n_2 - 1$

$f(n_1) = f(n_2)$

one-one :- let $f(n_1) = f(n_2)$

$n_1 - 1 = n_2 - 1$

$n_1 = n_2$

Onto :- let $x \in W$

then clearly $x+1 \in \mathbb{N}$

and $f(x+1) = x+\cancel{1}-\cancel{1} = x$

that means the mapping is onto

Hence $\exists$ one-one correspondence b/w $\mathbb{N}$ and W card $\mathbb{N}$ = card W

Q. $f: Z \to 2Z$

$f(n) = 2n$

well-defined :- $f(n) = 2n$

let $n_1 = n_2$

$2n_1 = 2n_2$

$f(n_1) = f(n_2)$

one-one :- $f(n_1) = f(n_2)$

$2n_1 = 2n_2$

$n_1 = n_2$

onto :- let $x \in 2Z$

If $\exists$ $n \in Z$ such that $f(n) = x$

i.e $2n = x$

$\Rightarrow n = x/2 \in Z$

so take $n = \frac{x}{2}$ so that $f(n) = f\left(\frac{x}{2}\right) = x$

$\Rightarrow f$ is onto

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Q. $f: \mathbb{Z} \to \mathbb{Z} | 2\mathbb{Z}$ = set of odd integer

$f(n) = 2n-1$

or

$f(n) = 2n+1$

* well defined : let $n_1 = n_2$

$2n_1 = 2n_2$

$2n_1 - 1 = 2n_2 - 1$

$f(n_1) = f(n_2)$

one-one :- $f(n_1) = f(n_2)$

$2n_1 - x = 2n_2 - x$

$2n_1 = 2n_2$

$n_1 = n_2$

onto :- let $x \in 2\mathbb{Z}-1$

T.P $\exists$ $n \in \mathbb{Z}$

$f(n) = x$

$2n+1 = x$

$n = \frac{x-1}{2}$

as $\frac{x-1}{2} \in \mathbb{Z}$

so it is onto

Q. $f: \mathbb{Z} \to \mathbb{N}$

$f(n) = \begin{cases} -2n+1 & \text{if } n=0 \text{ or } n \text{ is -ve} \\ 2n & \text{if } n \text{ is positive} \end{cases}$

$0 \to 1$ $\quad$ $1 \to 2$

$-1 \to 3$ $\quad$ $2 \to 4$

$-2 \to 5$ $\quad$ $3 \to 6$

$-3 \to 7$ $\quad$ $4 \to 8$

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well-defined:- let $x_1 = x_2$

case 1:- If x_1 & x_2 are negative integers or zero

$$-2x_1 = -2x_2$$
$$-2x_1 + 1 = -2x_2 + 1$$
$$f(x_1) = f(x_2)$$

Case 2:- If x_1 & x_2 are +ve

$$x_1 = x_2$$
$$2x_1 = 2x_2$$
$$f(x_1) = f(x_2)$$

one-one:- let $f(x_1) = f(x_2)$

case 1 :- If x_1, x_2 are neg. integers or 0 then

$$f(x_1) = f(x_2)$$
$$-2x_1 + 1 = -2x_2 + 1$$
$$-2x_1 = -2x_2$$
$$x_1 = x_2$$

Case 2:- If x_1, x_2 are +ve integers then

$$f(x_1) = f(x_2)$$
$$2x_1 = 2x_2$$
$$x_1 = x_2$$

Onto:- let $x \in N$. To find $n \in Z$ such that $f(n) = x$

case 1:- If x is odd

To find $n \in Z$ such that $f(n) = x$

$$-2n + 1 = x$$
$$2n = 1 - x$$
$$n = \frac{1-x}{2} \in Z$$

case 2:- when x is even

$$f(n) = x$$
$$2n = x, \Rightarrow n = x/2 \in Z$$

Mathematical Induction

1. Prove by MI that

$$1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Step I we'll prove that the result is true for $n=1$

L.H.S: $1^2 = 1$

R.H.S: $\frac{1(1+1)(2+1)}{6} \Rightarrow \frac{6}{6} = 1$

Step (2) Assume that the result is true for $n=k$

L.H.S: $1^2+2^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$

Step (3) To prove that result is true for $n=k+1$

$$= 1^2+2^2+\dots+k^2+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6} \Rightarrow \frac{(k+1)(2k^2+4k+3k+6)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6} \Rightarrow \frac{(k+1)(2k+3)(k+2)}{6}$$

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(2) Show that $2^n \times 2^n - 1$ is divisible by 3 $\forall n \in N$

Step I :- for $n=1$

$2^2 \times 2^n = 1 \Rightarrow 2 \times 2 - 1 = 3$ which is divisible by 3

Step II . Assume that $2^k \cdot 2^k - 1$ is divisible by 3

~~Step III~~ $2^k \cdot 2^k - 1 = 3m$ for sum +ve integer m.

$2^{2k} = 3m+1$

Step III take $n = k+1$

$2^n \times 2^{n+1} = 1 \Rightarrow 2^{k+1} \cdot 2^{k+1} - 1$

$\Rightarrow 2^{2k+2} - 1$

$\cdot 2^{2k} \cdot 2^2 - 1 = 2^{(3m+1)} \cdot 2^2 - 1$

$= 12m + 4 - 1$

$= 12m + 3$

$= 3(4m+1)$

which is divisible by 3

(3) Consider set Z of integer. Let $m \in Z$ s.t $m > 1$. we say that for two integer x, y; x is congruent to y modulo m if m divides $x - y$ (written as $x \equiv y \pmod{m}$. Prove that this is an equivalence relation on Z.

Proof :- Reflexive :-

relation is reflexive - if $x \equiv x \pmod{m}$

i.e if $m | x - x$
i.e if $m | 0$ which is true

so $x \equiv x \pmod m$

<u>Symm</u>
relation symm if $x \equiv y \pmod m$ then
$y \equiv x \pmod m$
i.e if $m | x - y$ then $m | y - x$
which is true.

<u>Transitive</u> : Let $x \equiv y \pmod m$ and $y \equiv z \pmod m$

$m | x - y$ and $m | y - z$
$x - y = mk$ and $y - z = mt$ for some Integer k
Such that adding these, we get
$x - \cancel{y} + \cancel{y} - z = mk + mt$
$\Rightarrow x - z = m(k+t)$
$\Rightarrow m | x - z \Rightarrow x \equiv z \pmod m$

~~relation is sym~~

<u>Well-Ordering property of positive Integers.</u>

Let S be a non-empty subset of set of +ve integers. Then S has a least element. In other words, ∃ an element $a \in S$ such that $a \le b \; \forall \; b \in S$

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Fundamental Theorem of Arithmetic

every Positive Integer can be written as product of power of finitely many primes, in a unique way (except for the order primes).

$20 = 2^2 \times 5$
$= 5 \times 2^2$

Division Algorithm: "Given any two integers a and b, with $b > 0$, there exist unique integers q and r satisfying $a = bq + r$, $0 \le r < b$
$a - bq = r$

The integer q and r are called quotient and remainder respectively.

$5 \overline{)24}$ 4
20
4 — r

Proof - $S = \{a - xb \mid x \text{ is an integer}, a - xb \ge 0\}$
firstly we show that $S \ne \phi$
as $b \ge 1 \Rightarrow |a| b \ge |a| + a$

$\Rightarrow a - (-|a| b) = \cancel{a + |a| b}$
$= a + |a| b \ge a + |a|$
≥ 0

so S is a non-empty subset of integers
so S has well-ordering property.
That means S has a least integer says
$r = a - qb$ for some integer q

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$\therefore a = bq + r$ and $r \geq 0$

T.P. :- $r < b$

let $r \geq b$

take $a - (q+1)b$ $= a - qb - b$

$= r - b$

$\therefore r - b \geq 0$

$\Rightarrow a - (q+1)b \in S$

$\Rightarrow r - b \in S$

also $r - b < r$

but r is smallest integer in S

so, $r - b \in S$ is contradiction

~~Hence,~~ Hence $r < b$

$a = bq + r, \quad 0 \leq r < b$

uniqueness:- let q_1, r_1 are also two integers such that $a = bq_1 + r_1, \ 0 \leq r_1 < b$

$a - a = bq + r - bq_1 - r_1$

$0 = b(q - q_1) + (r - r_1)$

$b(q - q_1) = -(r - r_1)$

$|b(q - q_1)| = |r - r_1|$

$|b||q - q_1| = |r - r_1|$ —— (1)

Also :- $0 \leq r < b$

$0 \leq r_1 < b$

$-b < -r \leq 0$

$-b < r - r_1 < b$

$\Rightarrow |r_1 - r_1| < b$

from (1)

$|b||q - q_1| < b$

$b|q-q_1| < b$

$|q-q_1| < 1$

but q_1 and q_2 are integers

$\Rightarrow q_2 - q_1 = 0$

$\Rightarrow q_2 = q_1$

also $r_1 = a - bq_1$

$= a - bq$

$= r$

$r_1 = r$

Divisibility :- An integer b is said to be divisible by an integer a $(\neq 0)$ if there exists some integer c such that $b = ac$

we write it as $a|b$

If a does not divisible b

$a \nmid b$

Theorem:- for integer a, b, c the following hold

(i) $a|0$, $1|a$, $a|a$

Proof:- $0 = a \cdot 0$

$\Rightarrow a|0$

$a = 1 \cdot a$

$= 1|a$

$a = a \cdot 1$

$= a|a$

(ii) $a|1$ iff $a = \pm 1$

let $a|1$

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$\Rightarrow 1 = a.c$ where $c \in \mathbb{Z}$
$\Rightarrow a = 1 = c$ or $a = -1 = c$
$\Rightarrow a = \pm 1$

Conversely let $a = 1$ or $a = -1$
then as $1 = 1.1$ and $1 = -1.(-1)$
$a/1$ and $-1/1$
$= a/1$

(III) a/b and c/d then ac/bd
$b = ak$ for some $k \in \mathbb{Z}$
$d = ct$ for some $t \in \mathbb{Z}$
$bd = ak.ct$
$bd = ac.kt$ (as $k, t \in \mathbb{Z} \Rightarrow kt \in \mathbb{Z}$)
$\Rightarrow ac/bd$

IV If a/b and b/c then a/c
$b = ak$ for some $k \in \mathbb{Z}$
$c = bt$ for some $t \in \mathbb{Z}$

$c = (ak).t$ (as $k, t \in \mathbb{Z}$
$c = akt$ $kt \in \mathbb{Z}$
$\Rightarrow a/c$

V a/b and b/a iff $b = \pm a$

as $a/b \Rightarrow b = ak$ for some $k \in \mathbb{Z}$
as $b/a \Rightarrow a = bt$ for some $t \in \mathbb{Z}$
$a = akt$
$a - akt = 0$
$a(1 - kt) = 0$
$a = 0$ or $1 - kt = 0$
$a = 0$ or $kt = 1$

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$a=0$ or $k=1$ or $k=-1$

If $a=0$

$b=ak=0$

Hence $b=a$

If $k=1$

$b=ak=a$

If $k=-1$

$b=-a$

Conversely :- let $b=\pm a$

If $b=a$

then $a|b$ and $b|a$

If $b=-a$

then $b=a(-1)$

$\Rightarrow a|b$

and $b=-a \Rightarrow a=-b$

$a=b(-1)$

$\therefore b|a$

VI If $a|b$ and $b\neq0$

then $|a|\leq|b|$

as $a|b$

$b=ak$ for some integer $k(\neq0)$

$\Rightarrow |b|=|ak|$

$\Rightarrow |b|=|a||k|$

as $k\neq0 \Rightarrow |k|\geq1$

So from (1)

$|b|\geq|a|$

VII If $a|b$ and $a|c$

then $a|(bx+cy)$ for arbitrary integer x and y

as $a|b \Rightarrow b=ak$ for some integer k

as $a|c \Rightarrow c=at$ for some integer t

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Now for arbitrary integers x, y

$bx + cy = (ak)x + (at)y$

$= akx + aty$

$= a(kx + ty)$ where $kx + ty \in \mathbb{Z}$

$a \mid (bx + cy)$

G.C.D (Greatest Common Divisor) or H.C.F (Highest common factor).

Def:- If a and b are two integers where at least one of them is non-zero. The GCD of a and b written as $\gcd(a, b)$ is a positive integer d such that

(1) $d \mid a$ and $d \mid b$

(2) If there is some integer c such that $c \mid a$ and $c \mid b$, then $c \leq d$

Result:- If $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$

Proof:- Let $d = \gcd(a, b)$ and $c = \gcd(b, r)$

$d \mid a$ and $d \mid b$

$\Rightarrow d \mid a$ and $d \mid bq$

$\Rightarrow d \mid (a - bq)$

$\Rightarrow d \mid r$ $\Rightarrow$ $\therefore$ so $d \mid b$ and $d \mid r$

but $c = \gcd(b, r)$ $\Rightarrow d \leq c$

Now, $c = \gcd(b, r)$

$\Rightarrow c \mid b$ and $c \mid r$ $\Rightarrow c \mid bq$ and $c \mid r$

$\Rightarrow c \mid r + qb$ $\Rightarrow$ $c \mid a$

also $c \mid b$

but $\gcd(a, b) = d$

$\Rightarrow c \leq d$

Hence $c = d$

so $\gcd(a, b) = \gcd(b, r)$

where $a = bq + r$